

Covariant Hamiltonian evolution in supersymmetric quantum systems

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ABSTRACT: We develop a general formalism for covariant Hamiltonian evolution of supersymmetric (field) theories by making use of the fact that these can be represented on the exterior bundle over their bosonic configuration space as generalized Dirac-Kähler systems of the form $(\mathbf{d} \pm \mathbf{d}^\dagger) |\psi\rangle = 0$. By using suitable deformations of the supersymmetry generators we find covariant Hamiltonians for target spaces with general gravitational and Kalb-Ramond field backgrounds and discuss their perturbation theory. As an example, these results are applied to the study of curvature corrections of superstring spectra for $\text{AdS}_3 \times S^3$ close to its pp-wave limit.

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1. Introduction

Supersymmetric quantum theory and differential (possibly non-commutative) geometry are in some sense two aspects of the same thing, as has been emphasized long ago in articles such as [1] and [2, 3].

This correspondence becomes however manifest only in the *Schrödinger representation* of quantum theory, where states are expressed as functionals over configuration space and operators act by (functional) multiplication and (functional) differentiation.¹ For field theoretic applications this representation is usually considered more awkward than the common Fock space representation, however it certainly has also advantages, at least on the conceptual level. In particular Hamiltonians (and Hamiltonian *constraints* for that matter) can be identified with (generalized) *Laplace operators* on configuration space, which makes manifest the connection between quantum (field) theory and the geometry of configuration space. Supersymmetric quantum (field) theory furthermore provides the corresponding *Dirac operators*. In [2, 3] it was stressed that the quantum theory provides thus nothing else but *spectral data* of configuration space.

Some illustrative examples of this correspondence have been made explicit in [5]. The quantum theory of a scalar field plus superpartner, as well as $N = 1$, $d = 4$ supergravity were represented on their respective configuration space by generalized Dirac-Kähler equations. Schematically these look like

$$(\mathbf{d} \pm \mathbf{d}^\dagger) |\psi\rangle = 0. \quad (1.1)$$

Here \mathbf{d} denotes the (generalized) exterior derivative on configuration space and \mathbf{d}^\dagger is its adjoint with respect to the Hodge inner product on differential forms over configuration space. The state $|\psi\rangle$ is a section of this exterior bundle, which itself must be regarded as the superspace over the original bosonic configuration space.

But also the superconformal constraints of Type II strings are of this Dirac-Kähler form, if \mathbf{d} is taken to be a generalized and deformed exterior derivative over loop space, the configuration space of the string. For superstrings in gravitational and Kalb-Ramond backgrounds this was perhaps first noticed in [6]. That the same holds true for all massless NS backgrounds and that the deformation can always be written as a similarity transformation $\mathbf{d} \rightarrow e^{-W} \mathbf{d} e^W$ is shown in [7]. There are indications that this statement indeed generalizes to all kinds of background fields [8].

The motivation for studying the Dirac-Kähler representation of supersymmetric systems, and of superstrings in particular, comes from its emphasis of the role of generalized Dirac operators in these theories. The program of Connes' Noncommutative Geometry [9] in the sense of a theory of spectral geometry shows that a great

¹For a list of references on field theory in Schrödinger representation see for instance [4].

deal of information is encoded and can be extracted from generalized Dirac operators. The relation between supersymmetric physics and Noncommutative Geometry has been particularly emphasized in [2, 3] and the identification of the superstring's worldsheet supercharge with a Dirac operator in a spectral triple has been used in [10, 11, 12] to study stringy symmetries and dualities. However, with the advent of the M=Matrix proposal [13] the emphasis on Noncommutative Geometry in the string theory literature has shifted from its *spectral* aspect, and hence the role of the Dirac operator, towards its *noncommutative* aspect and the role of the algebra (for instance [14]).

One purpose of this paper and its companion [7] is to demonstrate that an emphasis on the Dirac-Ramond operator of string theory, generalized to arbitrary backgrounds, allows to address current questions in string theory in an interesting and worthwhile way. In particular in this paper we shall try to address the general issue of computation of superstring spectra in nontrivial and not exactly solvable backgrounds from a perspective that puts the worldsheet supercharge and its role as a Dirac operator on loop space in the center of attention. We describe a covariant formalism for the construction of superstring Hamiltonians associated with arbitrary timelike Killing vectors of target space. These Hamiltonians are to be regarded as NSR-string analogues to the lightcone Hamiltonian of the Green-Schwarz string.

The latter has recently been used in the study of curvature corrections to string spectra on AdS backgrounds. Motivated by the insight [15] that the string spectrum on the pp-wave limit of such backgrounds corresponds directly to certain states in the dual field theory, as predicted by the AdS/CFT correspondence, several attempts are being made to extend this result to higher order corrections by perturbatively computing the spectrum on the string theory side [16, 17, 18].

In all these approaches lightcone gauge needs to be fixed. As always, this choice of gauge brings with it some simplifications but also a couple of technical subtleties [18]. In general it is restricted to backgrounds that possess a lightlike Killing vector. In general, the quantum theory of the Green-Schwarz string is poorly understood.

The purpose of the following discussion is to analyze the question whether it is possible to construct appropriate Hamiltonians and their perturbation theory not in the context of the Green-Schwarz string but in that of the covariantly quantized NSR superstring with manifest worldsheet supersymmetry. The first part of the paper, §2 (p.8), approaches this question in a general way by developing a covariant Hamiltonian perturbation theory for any supersymmetric systems which are governed by constraints of the Dirac-Kähler type. In the second part §3 (p.39) this general theory is applied to the superstring by making use of results given in [7]. As a first example application we test our perturbation theory in a well-understood context, namely the pp-wave limit of $\text{AdS}_3 \times \text{S}^3$, which can be regarded as a toy example for the more interesting $\text{AdS}_5 \times \text{S}^5$ case [16].

The key ideas of the perturbation theory developed here are the following:

Given a (pseudo-)Riemannian manifold (\mathcal{M}, g) the inhomogeneous differential forms $|\phi\rangle$ over it form an inner product space \mathcal{H} with respect to the Hodge inner product $\langle \cdot | \cdot \rangle$. We consider physical systems modeled on such a space of states and governed by constraints of the form

$$\mathbf{D}_{\pm}^{(A)} |\phi\rangle = 0, \quad (1.2)$$

where $\mathbf{D}_{\pm}^{(A)}$ are Dirac operators on \mathcal{H} , which are obtained from the ordinary Dirac(-Kähler) operators $\mathbf{D}_{\pm} = \mathbf{d} \pm \mathbf{d}^{\dagger}$ by a deformation

$$\begin{aligned} \mathbf{d}^{(A)} &:= A^{-1} \mathbf{d} A, \quad \mathbf{d}^{\dagger(A)} := (\mathbf{d}^{(A)})^{\dagger} \\ \mathbf{D}_{\pm}^{(A)} &:= \mathbf{d}^{(A)} \pm \mathbf{d}^{\dagger(A)}, \end{aligned} \quad (1.3)$$

where A is any invertible operator on \mathcal{H} . The relevance of these assumptions for string theory lies in the fact that the RNS superstring in various backgrounds can be rewritten this way when \mathcal{M} is taken to be loop space over spacetime.

After introducing analogous deformations of the form creation operators $\hat{c}^{\dagger\mu} := dx^{\mu} \wedge$ and the associated Clifford generators $\hat{\gamma}_{\pm} := \hat{c}^{\dagger} \pm \hat{c}$ by setting

$$\begin{aligned} \hat{c}^{\dagger(A)} &:= A^{\dagger-1} \hat{c}^{\dagger} A^{\dagger} \\ \hat{\gamma}_{\pm}^{(A)} &:= \hat{c}^{\dagger(A)} \pm \hat{c}^{(A)} \end{aligned} \quad (1.4)$$

it is easy to see that the Lie derivative operator \mathcal{L}_{v_0} along a timelike Killing vector v_0 of (\mathcal{M}, g) can be expressed as

$$\mathcal{L}_{v_0} = \frac{1}{4} \left(\left\{ \hat{\gamma}_{+}^{(A)}, \mathbf{D}_{-}^{(A)} \right\} - \left\{ \hat{\gamma}_{-}^{(A)}, \mathbf{D}_{+}^{(A)} \right\} \right). \quad (1.5)$$

It follows that the constraints (1.2) imply a Schrödinger equation

$$i\mathcal{L}_{v_0} |\phi\rangle = \mathbf{H}^{(A)} |\phi\rangle \quad (1.6)$$

of evolution along the parameter t_{v_0} , where the Hamiltonian is given by²

$$\begin{aligned} \mathbf{H}^{(A)} &:= \frac{i}{4} \left(\left[\hat{\gamma}_{-}^{(A)}, \mathbf{D}_{+}^{(A)} \right] - \left[\hat{\gamma}_{+}^{(A)}, \mathbf{D}_{-}^{(A)} \right] \right) \\ &= \frac{i}{2} \left(\hat{\gamma}_{-}^{(A)} \mathbf{D}_{+}^{(A)} - \hat{\gamma}_{+}^{(A)} \mathbf{D}_{-}^{(A)} \right) + i\mathcal{L}_0. \end{aligned} \quad (1.7)$$

For the deformations considered here there is a Krein space operator $\hat{\eta}$ (with $\hat{\eta}^{\dagger} = \hat{\eta}$ and $\hat{\eta}^2 = 1$) which serves to define a positive definite scalar product³ $\langle \cdot | \cdot \rangle_{\hat{\eta}} :=$

²In the context of classical electromagnetism, to which the present formalism also applies (*cf.* appendix C (p.73)), this operator is sometimes known as the Maxwell operator generating time evolution of the electromagnetic field.

³The term $\delta(t_{v_0})$ restricts integration to a hypersurfaces orthogonal to the flow lines of the Killing vector v_0 .

$\langle \cdot | \delta(t_{v_0}) \hat{\eta} | \cdot \rangle$ on physical states with respect to which $\mathbf{H}^{(A)}$ is hermitian:⁴

$$(\mathbf{H}^{(A)})^{\dagger_{\hat{\eta}}} := \hat{\eta}(\mathbf{H}^{(A)})^{\dagger} \hat{\eta} = \mathbf{H}^{(A)}. \quad (1.8)$$

This allows to perform quantum mechanical perturbation theory in a fully covariant framework. Even though no light cone gauge is required a particularly simple formula for the first order energy shift of a given state under a given perturbation of the constraints (1.2) is obtained when v_0 is of the form $v_0 = e^{\gamma} p + e^{-\gamma} k$ for p and k two lightlike Killing vectors: In the limit $\gamma \gg 1$ we find for the first order shift of the light cone energy associated with p the expression

$$e^{-\gamma} \langle \phi^{(0)} | \hat{\eta} (\mathbf{H}^{(A)})^{(1)} | \phi^{(0)} \rangle \rightarrow \langle \phi^{(0)} | \hat{\eta} i(\mathcal{L}_p)^{(1)} | \phi^{(0)} \rangle \quad (1.9)$$

(where the n -th order perturbation of an object O is written as $O^{(n)}$).

The structure of this paper is as follows:

The central results concerning covariant Hamiltonian evolution in supersymmetric systems are developed in §2 (p.8). First some existing material on supersymmetric quantum theory in differential geometric formulation is recalled in a coherent fashion in §2.1 (p.8), where we also elaborate on the general structure of deformations of the supersymmetry algebra and on the differential geometric meaning of the operators appearing in quantum SWZW models.

These facts are then used in §2.2 (p.21) to construct the general formalism for covariant Hamiltonian evolution in backgrounds with arbitrary metric and NS 2-form fields. Finally the associated perturbation theory is developed.

In §3 (p.39) these techniques are applied to the perturbative calculation of superstring spectra. First §3.1 (p.39) reviews the loop space formulation which puts the NSR string in the required Dirac-Kähler form. In §3.2 (p.42) basic data of the $\text{AdS}_3 \times S^3$ background as well as its pp-wave Penrose limit are listed, which are then inserted in §3.3 (p.45) into the perturbation formalism developed before. The obtained perturbative spectrum of strings in this scenario is then compared in §3.4 (p.47) to the exact result and the implications of the calculation are discussed in §3.5 (p.48).

Further details are given in the appendices:

Appendix A (p.53) collects various objects that play a role in the formulation of differential geometry in terms of operators on the exterior bundle, which is the technical basis for the formulation of supersymmetric quantum theory used here. Most of this material is elementary and mainly meant to set up notation and concepts,

⁴The Hamiltonian $\mathbf{H}^{(A)}$ needs furthermore to commute with t_{v_0} . For certain deformations this requires slight modifications of the general argument summarized here.

but also some not so widely known facts are derived and emphasized, which are crucial for the developments in §2 (p.8).

In appendix B (p.69) proofs are given which are omitted for the sake of brevity in the main text.

Appendix C (p.73) illustrates our formalism in terms of a well-known example to which it happens to apply, too, namely that of classical electromagnetism in differential form language.

Finally appendix D (p.75) lists some standard facts about Lie algebras that are needed for the discussion of SWZW models in §2.1.4 (p.17).

2. Covariant parameter evolution for supersymmetric quantum systems

2.1 On supersymmetric quantum theory in geometrical formulation

2.1.1 Introduction

Let the (pseudo-)Riemannian manifold (\mathcal{M}, g) be the configuration space of some physical system. A supersymmetric extension of this system has as configuration space the superspace \mathcal{SM} over \mathcal{M} , which can be identified with $\Omega^1(\mathcal{M})$, the 1-form bundle over \mathcal{M} . An arbitrary quantum state of the supersymmetric system is therefore a superfunction on $\Omega^1(\mathcal{M})$, which is an inhomogeneous differential form over \mathcal{M} , i.e. an element of $\Gamma(\Omega(\mathcal{M}))$, the space of sections of the total form bundle $\Omega(\mathcal{M}) = \bigoplus_{p=0}^D \Omega^p(\mathcal{M})$.

Before proceeding we briefly list some of the notation which will be used frequently in the following. The details are given in §A (p.53):

$\Gamma(\Omega(\mathcal{M}))$	the space of sections of the exterior bundle	
$\langle \cdot \cdot \rangle$	the Hodge inner product on $\Gamma(\Omega(\mathcal{M}))$	(A.2)
$\hat{c}^{\dagger\mu} = dx^\mu \wedge$	operator of exterior multiplication by dx^μ	(A.1)
$\hat{c}^\mu = dx^\mu \lrcorner$	operator of interior multiplication	(A.3)
$\hat{\gamma}_\pm^\mu = \hat{c}^{\dagger\mu} \pm \hat{c}^\mu$	the associated Clifford algebra generators	(A.4)
$\hat{\nabla}_\mu = \partial_\mu^c - \Gamma_\mu^\alpha{}_\beta \hat{c}^{\dagger\beta} \hat{c}_\alpha$	covariant derivative on $\Gamma(\Omega(\mathcal{M}))$	(A.23)(A.29)
$\mathbf{d} = \hat{c}^{\dagger\mu} \hat{\nabla}_\mu$	the exterior derivative on $\Gamma(\Omega(\mathcal{M}))$	(A.39)
$\mathbf{d}^\dagger = -\hat{c}^\mu \hat{\nabla}_\mu$	its adjoint with respect to $\langle \cdot \cdot \rangle$	(A.47)
$\mathbf{D}_\pm = \mathbf{d} \pm \mathbf{d}^\dagger$	the associated Dirac operators	(A.51)
$\mathcal{L}_v = \{\mathbf{d}, v^\mu \hat{c}_\mu\}$	Lie derivative along v	(A.69)
$\hat{\eta}$	a Krein space operator	(A.20)(2.81)
$\langle \cdot \cdot \rangle_{\hat{\eta}} = \langle \cdot \hat{\eta} \cdot \rangle$	a scalar product on $\Gamma(\Omega(\mathcal{M}))$	(A.21)

The systems of interest here will have semi-Riemannian configuration space metric g and be governed by sets of equations that are generalizations of

$$\mathbf{D}_+ \omega = 0 = \mathbf{D}_- \omega, \quad \omega \in \Gamma(\Omega(\mathcal{M})) \quad (2.1)$$

(cf. (A.51)). In the case that $\Delta = \mathbf{D}_+^2$ (cf. (A.52)) is taken as a generator of “time”-translations in the system’s parameter space the relations

$$\{\mathbf{D}_+, \mathbf{D}_+\} = 2\Delta \quad (2.2)$$

may be regarded as the $(N=1)$ -supersymmetry algebra in 1 dimension. Similarly

$$\{\mathbf{D}^i, \mathbf{D}^j\} = 2\delta^{ij}\Delta, \quad (2.3)$$

where $i, j \in \{1, 2\}$ and $\mathbf{D}^1 := \mathbf{D}_+$, $\mathbf{D}^2 := i\mathbf{D}_-$, is the 1-dimensional supersymmetry algebra with $N = 2$, which is of course equivalent to

$$\begin{aligned}\{\mathbf{d}, \mathbf{d}\} &= 0 \\ \{\mathbf{d}^\dagger, \mathbf{d}^\dagger\} &= 0 \\ \{\mathbf{d}, \mathbf{d}^\dagger\} &= \Delta.\end{aligned}\tag{2.4}$$

This algebra gives us 1+0 dimensional supersymmetric field theory, i.e. supersymmetric quantum mechanics.

Is there a deformation of \mathbf{d} , \mathbf{d}^\dagger that turns this algebra into the 2-dimensional ($N = 1$)-algebra? Recall the central observation from the last part of [1]:

When choosing, as usual, the unitary representation of the Clifford algebra $\text{Cl}(1, 1)$ given by

$$\begin{aligned}\gamma_{AB}^0 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \gamma_{AB}^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\end{aligned}\tag{2.5}$$

and supercharges Q that are real,

$$\bar{Q} = Q^T \gamma^0,\tag{2.6}$$

then the “ $QQ = P$ ” bracket looks like

$$\begin{aligned}\{Q_A, Q_B\} &= -(\gamma^\mu \gamma^0)_{AB} P_\mu \\ &= \begin{bmatrix} P_0 + P_1 & 0 \\ 0 & P_0 - P_1 \end{bmatrix}_{AB}.\end{aligned}\tag{2.7}$$

In terms of the linear combinations

$$\begin{aligned}d_k &:= \frac{1}{\sqrt{2}} (Q_1 - iQ_2) \\ d_k^* &:= \frac{1}{\sqrt{2}} (Q_1 + iQ_2)\end{aligned}\tag{2.8}$$

this is equivalent to

$$\begin{aligned}\{d_k, d_k\} &= P_1 \\ \{d_k^*, d_k^*\} &= P_1 \\ \{d_k, d_k^*\} &= P_0.\end{aligned}\tag{2.9}$$

This is almost of the form (2.4), except that the $\{\mathbf{d}, \mathbf{d}\}$ and $\{\mathbf{d}^\dagger, \mathbf{d}^\dagger\}$ brackets pick up a non-zero value equal to the generator P_1 of spatial translations. One way to realize this deformation is the following:

Deformations by Killing vectors. In the presence of a Killing vector $k = k^\mu \partial_\mu$, one can consider a deformation \mathbf{d}_k of the exterior derivative defined by

$$\mathbf{d}_k := \mathbf{d} + i\hat{c}_\mu k^\mu. \quad (2.10)$$

The adjoint operator is then

$$\mathbf{d}_k^\dagger := \mathbf{d}^\dagger - i\hat{c}^\dagger_\mu k^\mu. \quad (2.11)$$

By the definition of the Lie-derivative (A.69) one finds

$$\mathbf{d}^2 = i\mathcal{L}_k, \quad (2.12)$$

and, since k is Killing, by (A.79) also

$$\mathbf{d}^{\dagger 2} = i\mathcal{L}_k. \quad (2.13)$$

Defining

$$\begin{aligned} \mathbf{D}_{k,\pm} &= \mathbf{d}_k \pm \mathbf{d}_k^\dagger \\ &= \gamma^\mu_{\mp} \left(\hat{\nabla}_\mu \mp i k_\mu \right) \end{aligned} \quad (2.14)$$

one has, with $A, B \in \{+, -\}$ and $s_\pm := \pm 1$,

$$\{\mathbf{D}_{k,A}, \mathbf{D}_{k,B}\} = 2\delta_{AB} (s_A \Delta_k + i\mathcal{L}_k), \quad (2.15)$$

where the deformed Laplace-Beltrami operator is

$$\begin{aligned} \Delta_k &:= \{\mathbf{d}_k, \mathbf{d}_k^\dagger\} \\ &= \Delta + k^2 + i \left(\{\mathbf{d}^\dagger, \hat{c}_\mu k^\mu\} - \{\mathbf{d}, \hat{c}^\dagger_\mu k^\mu\} \right) \\ &= \Delta + k^2 - i(\partial_{[\mu} k_{\nu]}) \left(\hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu} + \hat{c}^\mu \hat{c}^\nu \right). \end{aligned} \quad (2.16)$$

Note that the deformed exterior differential operators still satisfy the duality relation (A.46):

$$\mathbf{d}_k^\dagger = -\bar{\star} \mathbf{d}_k \bar{\star}. \quad (2.17)$$

This gives us the algebra of $d = 2$, $N = 1$ supersymmetry, which is necessary to describe the manifestly worldsheet supersymmetric string. It is now of interest how the generators of this algebra may be deformed in order to incorporate the effect of various background fields, without affecting the structure of the algebra itself.

2.1.2 Deformations of the supersymmetry generators.

The construction of supersymmetric quantum theories usually involves choosing a bosonic Lagrangian, replacing its fields with appropriate superfields, and integrating out the Grassmannian variables to obtain the supersymmetric Lagrangian of the component fields, which may finally be quantized. An alternative way to obtain new supersymmetric quantum theories, which shall be studied here, is to pick a given one (for instance a simple, free theory) and then deform its symmetry generators (for instance so as to introduce interaction and potentials) in a way that preserves the supersymmetry algebra. When working in the Schrödinger representation this may radically reduce the computational effort and increase transparency, as will be demonstrated here.

Furthermore, more importantly for the purposes of §2.2 (p.21), such deformations of the supersymmetry generators allow to deform other operators analogously such that results derived in the undeformed case can be rather straightforwardly adapted to the deformed case. This will be essential for the construction of the covariant Hamiltonian for b -field backgrounds in §2.2.3 (p.27).

Below it is shown that this strategy involves a generalization of the deformations already considered in the first part of [1]. Applications of this method to the study of actual physical systems have been rare, one example being [19] [5] (and references given there), where the method is applied to the study of supersymmetric quantum cosmology. In [7] and §3.1 (p.39) it is shown that it is also applied with some profit to the fundamental string.

We start by discussing deformations of the 1-dimensional supersymmetry algebra:

The case $D = 1, N = 2$. Recall from (2.4) that the $D = 1, N = 2$ supersymmetry algebra may be represented by operators \mathbf{d} and \mathbf{d}^\dagger (usually, but not necessarily, similar or equal to the exterior derivative and co-derivative), which satisfy

$$\begin{aligned}\{\mathbf{d}, \mathbf{d}\} &= 0 \\ \{\mathbf{d}^\dagger, \mathbf{d}^\dagger\} &= 0 \\ \{\mathbf{d}, \mathbf{d}^\dagger\} &= \Delta, \end{aligned} \tag{2.18}$$

as well as

$$(\mathbf{d})^\dagger = \mathbf{d}^\dagger \tag{2.19}$$

and therefore

$$\Delta^\dagger = \Delta. \tag{2.20}$$

Given any such an algebra, we are now looking for a 1-parameter family of algebra homomorphisms h_ϵ , $\epsilon \in \mathbb{R}$, which are continuously connected to the identity (i.e. h_0 is the identity operation) and which map these operators to

$$\begin{aligned}\mathbf{d}^\epsilon &:= h_\epsilon(\mathbf{d}) \\ \mathbf{d}^{\dagger\epsilon} &:= h_\epsilon(\mathbf{d}^\dagger) \\ \Delta^\epsilon &:= h_\epsilon(\Delta) ,\end{aligned}\tag{2.21}$$

in a way that preserves the relations (2.18) and (2.19). It is very easy to see which kinds of h_ϵ are possible:

By assumption of continuity we have

$$\mathbf{d}^\epsilon = \mathbf{d} + \epsilon \mathbf{X} + \mathcal{O}(\epsilon^2) ,\tag{2.22}$$

where \mathbf{X} is some operator to be determined. The algebra requires that

$$\begin{aligned}0 &= (\mathbf{d}^\epsilon)^2 \\ &= \epsilon \{\mathbf{d}, \mathbf{X}\} + \mathcal{O}(\epsilon^2) ,\end{aligned}\tag{2.23}$$

and therefore that \mathbf{d} anticommutes with its first order deformation:

$$\{\mathbf{d}, \mathbf{X}\} = 0 .\tag{2.24}$$

Since \mathbf{d} is nilpotent \mathbf{X} is locally “exact”

$$\mathbf{X} = [\mathbf{d}, \mathbf{W}] ,\tag{2.25}$$

where \mathbf{W} is any even graded operator. Assuming that \mathbf{X} is of this form we have

$$\begin{aligned}\frac{d}{d\epsilon} \mathbf{d}^\epsilon &= [\mathbf{d}, \mathbf{W}] \\ \Rightarrow \mathbf{d}^\epsilon &= \exp(-\epsilon \mathbf{W}) \mathbf{d} \exp(\epsilon \mathbf{W}) .\end{aligned}\tag{2.26}$$

We call

$$\mathbf{A} := \exp(\mathbf{W})\tag{2.27}$$

the deformation operator. The other deformed operators follow from this by

$$\begin{aligned}\mathbf{d}^{\dagger\epsilon} &:= (\mathbf{d}^\epsilon)^\dagger \\ &= \exp(\epsilon \mathbf{W}^\dagger) \mathbf{d}^\dagger \exp(-\epsilon \mathbf{W}^\dagger) \\ \Delta^\epsilon &:= \{\mathbf{d}^\epsilon, \mathbf{d}^{\dagger\epsilon}\} .\end{aligned}\tag{2.28}$$

Note that if \mathbf{W} is antihermitian the deformation is a pure gauge transformation.

Examples.

1. One example is the famous special case where $\mathbf{W} = W$ is the operator of multiplication by the real function W , which has been used in [1] to study Morse theory. If W were taken to be purely imaginary, then \mathbf{W} would be anti-hermitian and hence correspond to a pure phase shift symmetry that could be gauged away.
2. Consider the operators

$$\begin{aligned} \mathbf{d}^0 &= \hat{c}^{\dagger a} \partial_a \\ \mathbf{d}^{\dagger 0} &= -\hat{c}^a \partial_a \end{aligned} \quad (2.29)$$

on flat space (\mathcal{M}, η) (where η is the flat metric). Now pick a non-trivial metric g which in the ∂_a -basis satisfies $\det(g) = 1$ and pick (locally) an associated vielbein e^μ_a . Now there is an invertible linear operator \mathbf{A} defined by

$$\mathbf{A} \hat{c}^{\dagger a_1} \dots \hat{c}^{\dagger a_p} |1\rangle := e^{\mu=a_1}_{b_1} \hat{c}^{\dagger b_1} \dots e^{\mu=a_p}_{b_p} \hat{c}^{\dagger b_p} |1\rangle . \quad (2.30)$$

In fact, when we regard e^μ_a as a matrix e and let $\ln e$ be the logarithm of that matrix, then \mathbf{A} can be written in the form (2.27) as

$$\mathbf{A} = \exp\left(\hat{c}^\dagger \cdot (\ln e)^T \cdot \hat{c}\right) . \quad (2.31)$$

A little reflection shows that

$$\mathbf{d} = \mathbf{A} \mathbf{d}^0 \mathbf{A}^{-1} \quad (2.32)$$

is the operator representation of the exterior derivative on (\mathcal{M}, g) and hence

$$\mathbf{d}^\dagger = \mathbf{A}^{-1} \mathbf{d}^{\dagger 0} \mathbf{A}^\dagger \quad (2.33)$$

is its adjoint. (Of course \mathbf{d} as an abstract operator is independent of the metric on \mathcal{M} , but its representation in terms of operators $\hat{c}^{\dagger a}$ and ∂_a is not. Compare §A.2 (p.55).)

This way we can understand the metric field on the manifold as inducing a deformation of the supersymmetry generators of flat space. This will be seen to be a general phenomenon. In §2.1.3 (p.15) it is shown how similarly a Kalb-Ramond field background is represented by an exponential deformation operator.

3. The above restriction to $\det(g) = 1$ ensures that the inner product $\langle \cdot | \cdot \rangle$ and hence the adjoint operation $(\cdot)^\dagger$ itself receives no deformation (this follows from equations (A.2) and (A.37) that are given in the appendix). Alternatively one

can allow a conformal factor $e^{2\phi}$ but still keep the undeformed Hodge inner product. This describes a dilaton background:

$$\mathbf{A} = \exp(\phi \hat{c}^{\dagger\mu} \hat{c}_\mu) = \exp(\phi \hat{N}) . \quad (2.34)$$

(If instead the inner product is accordingly modified this \mathbf{A} induces an ordinary conformal transformation. This is discussed in §A.6 (p.66))

The example of central importance for the following is the case $\mathbf{A} = \exp(\frac{1}{2} b_{\mu\nu} \hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu})$, which induces a Kalb-Ramond field background. This is discussed in more detail for the $D = 2$ supersymmetry algebra in §2.1.3 (p.15).

Together with the metric and dilaton backgrounds discussed above this shows that all the massless NS-NS backgrounds of the superstring find their natural realization in terms of deformations of the supersymmetry generators.

The results for the 1-dimensional supersymmetry algebra straightforwardly carry over to two dimensions:

The case $D = 2, N = 1$. Now the supersymmetry algebra looks like (*cf.* (2.12), (2.13), (2.16))

$$\begin{aligned} \mathbf{d}_k^2 &= i\mathcal{L}_k \\ \mathbf{d}_k^{\dagger 2} &= i\mathcal{L}_k \\ \{\mathbf{d}_k, \mathbf{d}_k^\dagger\} &= \Delta_k . \end{aligned} \quad (2.35)$$

We shall restrict attention to homomorphisms h_ϵ that leave the element \mathcal{L}_k invariant⁵:

$$h_\epsilon(\mathcal{L}_k) \stackrel{!}{=} \mathcal{L}_k , \quad \forall \epsilon . \quad (2.36)$$

The analysis then closely parallels that of the $(D = 1, N = 2)$ -case:

Setting again

$$\mathbf{d}_k^\epsilon = \mathbf{d}_k + \epsilon \mathbf{X} + \mathcal{O}(\epsilon^2) \quad (2.37)$$

one obtains from

$$\begin{aligned} i\mathcal{L}_k &\stackrel{!}{=} (\mathbf{d}_k^\epsilon)^2 \\ &= i\mathcal{L}_k + \epsilon \{\mathbf{d}_k, \mathbf{X}\} + \mathcal{O}(\epsilon^2) \end{aligned} \quad (2.38)$$

⁵When applied to the string, \mathcal{L}_k will be the generator of reparameterizations along the string. Because the string must be reparameterization invariant in any background this generator must be preserved by the deformation.

the already familiar condition

$$\{\mathbf{d}_k, \mathbf{X}\} = 0. \quad (2.39)$$

But now \mathbf{d}_k is nilpotent only modulo \mathcal{L}_k , so this is solved in analogy with (2.25) by setting

$$\mathbf{X} = [\mathbf{d}_k, \mathbf{W}] \quad (2.40)$$

subject to the condition that

$$[\mathcal{L}_k, \mathbf{W}] = 0. \quad (2.41)$$

As before, the family of homomorphisms is therefore given by

$$\begin{aligned} \mathbf{d}_k^\epsilon &:= \exp(-\epsilon \mathbf{W}) \mathbf{d}_k \exp(\epsilon \mathbf{W}) , & [\mathbf{W}, \mathcal{L}_k] &= 0 \\ \mathbf{d}_k^{\dagger\epsilon} &:= \exp(\epsilon \mathbf{W}^\dagger) \mathbf{d}_k^\dagger \exp(-\epsilon \mathbf{W}^\dagger) \\ \mathcal{L}_k^\epsilon &:= \mathcal{L}_k \\ \Delta_k^\epsilon &:= \{\mathbf{d}_k^\epsilon, \mathbf{d}_k^{\dagger\epsilon}\} . \end{aligned} \quad (2.42)$$

Note that

$$\begin{aligned} \mathbf{d}_k^\epsilon &= \mathbf{d}^\epsilon + h_\epsilon (ik_\mu \hat{c}^\mu) \\ \mathbf{d}_k^{\dagger\epsilon} &= \mathbf{d}^{\dagger\epsilon} - h_\epsilon (ik_\mu \hat{c}^{\dagger\mu}) . \end{aligned} \quad (2.43)$$

Such deformations of the $D = 2$ $N = 1$ supersymmetry algebra will play an important role in the following constructions. We will show that choosing \mathbf{W} to be a 2-form (a “ b -field”) gives the supersymmetry constraints associated with a Kalb-Ramond field background. The fact that these relatively complicated constraints can be obtained from algebraically simple deformations of the form (2.42) will make it possible to systematically generalize results pertaining to vanishing 2-form backgrounds to non-vanishing 2-form backgrounds. In particular this will allow us to adapt the construction of the Hamiltonian generator for pure metric backgrounds derived in §2.2.1 (p.21) to that for g - and b -field backgrounds in §2.2.3 (p.27). This task would have been rather unfeasible in terms of the complicated expanded form of the supersymmetry generators (see below).

2.1.3 Deformation by background B -field.

In this section it is shown how a Kalb-Ramond background gives rise to a deformation as discussed above.

Consider the case where on \mathcal{M} there is, in addition to the metric g (admitting the Killing vector k) an antisymmetric 2-form field

$$b := \frac{1}{2} b_{\mu\nu} dx^\mu \wedge dx^\nu \quad (2.44)$$

with field strength

$$h_{\mu\nu\rho} := (\mathbf{d}b)_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}. \quad (2.45)$$

In order to couple this background field to our system (2.35), it is natural to set in (2.42)

$$\mathbf{W}^{(b)} := \frac{1}{2}b_{\mu\nu}\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}. \quad (2.46)$$

For this choice one finds

$$\begin{aligned} \mathbf{d}_k^{\epsilon=1} &:= \mathbf{d}_k^{(b)} = \mathbf{d}_k + \frac{1}{6}\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}\hat{c}^{\dagger\rho}h_{\mu\nu\rho} + ik^\mu b_{\mu\nu}\hat{c}^{\dagger\nu} \\ &= \hat{c}^{\dagger\mu}\hat{\nabla}_\mu + \frac{1}{6}\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}\hat{c}^{\dagger\rho}h_{\mu\nu\rho} + ik^\mu (g_{\mu\nu}\hat{c}^\nu + b_{\mu\nu}\hat{c}^{\dagger\nu}) \\ \mathbf{d}_k^{\dagger\epsilon=1} &:= \mathbf{d}_k^{\dagger(b)} = \mathbf{d}_k^\dagger - \frac{1}{6}\hat{c}^\mu\hat{c}^\nu\hat{c}^\rho h_{\mu\nu\rho} - ik^\mu b_{\mu\nu}\hat{c}^\nu \\ &= -\hat{c}^\mu\hat{\nabla}_\mu - \frac{1}{6}\hat{c}^\mu\hat{c}^\nu\hat{c}^\rho h_{\mu\nu\rho} - ik^\mu (g_{\mu\nu}\hat{c}^{\dagger\nu} + b_{\mu\nu}\hat{c}^\nu). \end{aligned} \quad (2.47)$$

The first part of these expressions, the one coming from the deformation of the exterior derivative itself, was already considered in [3] (p. 25) as an example for a supersymmetric quantum theory involving torsion. We here note that the b -field deformation of the full $D = 2$ supersymmetry algebra (2.10) and (2.11) in addition gives the terms proportional to k^μ on the right of (2.47). It turns out that these are precisely the terms needed to identify the generators in (2.47) with the supersymmetry generators of the $D = 2$, $N = 1$ nonlinear supersymmetric sigma model which describes superstring propagation in the respective b -field background, *cf.* §3.1 (p.39). We thus have found an algebraic way to derive the constraints of the NSR superstring in gravitational and Kalb-Ramond backgrounds. Knowledge of the deformation operator $\exp(\frac{1}{2}b_{\mu\nu}\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu})$ allows us to algebraically relate these constraints to the ordinary \mathbf{d}_k and \mathbf{d}_k^\dagger operators.

In order to further analyze the result (2.47) note that the corresponding Dirac operators are

$$\mathbf{D}_{k\mp}^{(b)} = \hat{\gamma}_\pm^\mu (\hat{\nabla}_\mu^{(b)} - i(b_{\mu\nu} \mp g_{\mu\nu})k^\nu) - \frac{1}{12}h_{abc}\hat{\gamma}_\pm^a\hat{\gamma}_\pm^b\hat{\gamma}_\pm^c. \quad (2.48)$$

Here we have identified $\hat{\nabla}_\mu^{(b)}$ as a deformation of the covariant derivative operator

$$\begin{aligned} \hat{\nabla}_\mu^{(b)} &:= \partial_\mu + \frac{1}{4}\omega_{\mu ab}^+\hat{\gamma}^{a+}\hat{\gamma}^{b+} - \frac{1}{4}\omega_{\mu ab}^-\hat{\gamma}^{a-}\hat{\gamma}^{b-} \\ &= \hat{\nabla}_\mu + \frac{1}{4}h_{\mu ab}(\hat{e}^{\dagger a}\hat{e}^{\dagger b} + \hat{e}^a\hat{e}^b), \end{aligned} \quad (2.49)$$

which involves connections with torsion $\pm\frac{1}{2}h$

$$\omega_{abc}^\pm := \omega_{abc} \pm \frac{1}{2}h_{abc} \quad (2.50)$$

(cf. §A.3 (p.59) and §A.5 (p.65)), and which acts on the Clifford algebras $\hat{\gamma}^\pm$ as the covariant derivative associated with the connections with torsion ω^\pm , respectively:

$$[\nabla_\mu^{(b)}, v_a \hat{\gamma}^{a\pm}] = (\nabla_\mu^\pm v_a) \hat{\gamma}^{a\pm}. \quad (2.51)$$

Its commutators give the torsion deformed curvature operator (cf. (A.28))

$$\begin{aligned} \mathbf{R}_{\mu\nu}^{(h)} &:= [\hat{\nabla}_\mu^{(b)}, \hat{\nabla}_\nu^{(b)}] \\ &= \mathbf{R}_{\mu\nu} + \frac{1}{8}(\nabla_{[\mu} h_{\nu]ab}) (\hat{e}^{\dagger a} \hat{e}^{\dagger b} + \hat{e}^a \hat{e}^b) + \frac{1}{4} h_{\mu ac} h_{\nu}{}^c{}_b \hat{e}^{\dagger a} \hat{e}^b. \end{aligned} \quad (2.52)$$

This expression vanishes iff $\frac{1}{2}h$ is the parallelizing torsion (see (A.108) in appendix A.5 (p.65)) and $\nabla_{[\mu} h_{\nu]ab} = 0$. This is of course true if $h_a{}^b{}_c$ are the structure constants of a group manifold, which is an interesting special case to which we now turn:

2.1.4 SWZW models

For \mathcal{M} a Lie group manifold and $h = db$ twice its parallelizing torsion the above construction reduces to that of super Wess-Zumino-Novikov-Witten models. These are of course well known, but because SWZW models will play an important role as exactly solvable backgrounds from which our perturbation theory may proceed and since we will need the special representation (2.63), to be derived below, of the currents in terms of Lie derivatives on spinors, this section spells out some aspects of SWZW models in terms of the formalism used here.⁶

Another purpose of this section is to put the general construction of Hamiltonian generators in §2.2.1 (p.21) into perspective: As discussed below (see (2.61) and (2.63)) the anticommutator of the Dirac operators (2.48) with the Clifford generators associated with the invariant vielbein field e_a of the group manifold gives the “total current” operators, which are, however, essentially (up to a spurious term proportional to k^μ) Lie derivative operators along e_a . Accordingly the respective *commutator* gives the associated “Hamiltonian” (by the general scheme that will be discussed in §2.2.1 (p.21)). Except for the spurious term this is hence already almost what we are looking for. The constructions in §2.2.3 (p.27) may therefore also be regarded as a generalization of the concept of “currents” on group manifolds to more general backgrounds.

Equation (2.51) shows that a special case of high symmetry is one where there is a b -field with field strength h and a metric g such that two vielbein fields e_a^\pm exist, which are parallel with respect to the connections with torsion:

$$\nabla_\mu^\pm e_a^\pm = 0. \quad (2.53)$$

⁶A standard text on the ordinary 2D WZW model is [20]. The original supersymmetric extension of the WZW model was given in [21]. WZW models with extended supersymmetry are discussed in [22] and [23]. We mostly follow the treatment in [12].

According to a general fact about Lie groups (see (D.18) in appendix D (p.75)), this is true when g is the Killing metric on a group manifold and h_{abc} is proportional to the structure constants of that group:

$$\left[\nabla_\mu^{(b)}, \left(e_a^\pm \right)^\mu \hat{\gamma}_{\mu\pm} \right] = 0. \quad (2.54)$$

Note that, by (D.19), there exists a 3-form $h_{\mu\nu\lambda}$ such that

$$\omega^\pm[e^\pm]_{abc} = 0. \quad (2.55)$$

Inserting this into (2.49) gives the b -deformed covariant derivative operator

$$e_a^{\sigma\mu} \hat{\nabla}_\mu^{(b)} = e_a^{\sigma\mu} \left(\partial_\mu^\sigma - \sigma \frac{1}{4} \omega^{-\sigma} [e^\sigma]_{\mu bc} \hat{\gamma}_{-\sigma}^b \hat{\gamma}_{-\sigma}^c \right), \quad (2.56)$$

where $\sigma = +1$ or $\sigma = -1$ and ∂_μ^σ is the partial derivative operator that commutes with $e_a^{\sigma\mu} \hat{\gamma}_\mu$ (but, in general, not with $e_a^{-\sigma\mu} \hat{\gamma}_\mu$), cf. (A.31).)

This expression makes it manifest that this covariant derivative commutes with all the Clifford generators associated with the vielbein fields e^\pm :

$$\left[\hat{\nabla}_\mu^{(b)}, e_a^{\sigma\mu} \hat{\gamma}_\sigma^\mu \right] = 0. \quad (2.57)$$

Because of the relation

$$\left[e_a^{\sigma\mu} \hat{\nabla}_\mu^{(b)}, e_b^{\sigma\nu} \hat{\nabla}_\nu^{(b)} \right] = f_a^{\ c} e_c^{\sigma\mu} \hat{\nabla}_\mu^{(b)} \quad (2.58)$$

$$\begin{aligned} \left[e_a^{\sigma\mu} \hat{\nabla}_\mu^{(b)}, e_b^{-\sigma\nu} \hat{\nabla}_\nu^{(b)} \right] &= \left[e_a^\sigma, e_b^{-\sigma} \right] \\ &= 0 \end{aligned} \quad (2.59)$$

it now follows that the \pm -components of the model completely decouple: First of all we have

$$\begin{aligned} \left\{ \mathbf{D}_{k,\sigma}^{(b)}, e_a^{\sigma a} \hat{\gamma}_\sigma^\mu \right\} &= 0 \\ \left[\mathbf{D}_{k,\sigma}^{(b)}, e_a^{\sigma\mu} \hat{\nabla}_\mu^{(b)} \right] &= 0. \end{aligned} \quad (2.60)$$

The remaining non-vanishing anticommutator defines the “total currents” which are hence the superpartners of the fermions $\hat{\gamma}_\sigma^a$:

$$\begin{aligned} J_a^\sigma &:= \left\{ \mathbf{D}_{k,-\sigma}^{(b)}, \frac{\sigma}{2} e_a^{\sigma\mu} \hat{\gamma}_{\mu,\sigma} \right\} \\ &= J_a^{\text{bos}\sigma} + J_a^{\text{fer}\sigma}, \end{aligned} \quad (2.61)$$

where the bosonic currents J^{bos} and the fermionic currents J^{fer} are defined by

$$\begin{aligned} J_a^{\text{bos}\sigma} &:= e_a^{\sigma\mu} \left(\hat{\nabla}_\mu^{(b)} - i (b_{\mu\nu} - \sigma g_{\mu\nu}) k^\nu \right) \\ J_a^{\text{fer}\sigma} &:= -e_a^{\sigma\mu} \frac{1}{4} h_{\mu bc} \hat{\gamma}_\sigma^b \hat{\gamma}_\sigma^c \\ &= \sigma e_a^{\sigma\mu} \frac{1}{2} \omega[e^\sigma]_{\mu bc} \hat{\gamma}_\sigma^b \hat{\gamma}_\sigma^c. \end{aligned} \quad (2.62)$$

Using (2.56) and (2.62) one finds that the total current is, up to the k -dependent term, the Lie derivative operator (*cf.* (A.71)) along e_a^σ :

$$J_a^\sigma = \underbrace{e_a^{\sigma\mu} \left(\partial_\mu^\sigma + \frac{1}{2} \omega[e^\sigma]_{\mu bc} \left(\hat{\gamma}_+^b \hat{\gamma}_+^c - \hat{\gamma}_-^b \hat{\gamma}_-^c \right) \right)}_{=\mathcal{L}_{e_a^\sigma}} - i e_a^{\sigma\mu} (b_{\mu\nu} \mp g_{\mu\nu}) k^\nu. \quad (2.63)$$

In particular, since all the e_a^σ are Killing vectors, this Lie derivative operator splits into two Lie derivative operators on the \pm -spinors, as shown in (A.91) of §A.3 (p.59).

This is what should be compared with the general expression (2.128) of the Lie derivative in terms of anticommutators of fermions with the supercharges that is derived below in §2.2.3 (p.27). In fact, in the case where the spurious term vanishes

$$e_a^{\sigma\mu} (b_{\mu\nu} \mp g_{\mu\nu}) = 0 \quad (2.64)$$

for some index a , the covariant Hamiltonian constructed there is precisely the Hamiltonian associated with the time parameter flowing along e_a^σ .

We have emphasized the applicability of the present formalism to the superstring. But it should be stressed that it is in fact more general. Indeed we have not even specified the precise form of the Killing vector k , yet. For any such k we get from (D.11) for the fermionic currents the commutation relations

$$\begin{aligned} [J_a^{\text{fer}\sigma}, J_b^{\text{fer}\sigma}] &= f_a^c{}_b J_c^{\text{fer}\sigma} \\ [J_{a^\pm}^{\text{fer}\pm}, J_{b^\mp}^{\text{fer}\pm}] &= 0, \end{aligned} \quad (2.65)$$

i.e. a representation of the Lie algebra. Furthermore we generally get for the b -deformed Laplace-Beltrami operator on SWZW backgrounds the simple expression

$$\left(\mathbf{D}_{-\sigma}^{(b)} \right)^2 = \sigma \left(g^{ab} \hat{\nabla}_a^{(b)} \hat{\nabla}_b^{(b)} - \frac{1}{12} g^\nu{}_\nu d \right), \quad (2.66)$$

which is, up to a sign and a scalar shift, the quadratic Casimir of the group. This operator manifestly commutes with the fermions

$$\left[\left(\mathbf{D}^{(b)} \right)^2, e_{a\mu}^\sigma \hat{\gamma}_\sigma^\mu \right] = 0 \quad (2.67)$$

from which it follows that the total currents commute with $\mathbf{D}_\pm^{(b)}$:

$$[\mathbf{D}_{-\sigma}^{(b)}, J_a^\sigma] = 0. \quad (2.68)$$

However the bosonic analog of (2.65), namely

$$\begin{aligned} [J_a^{\text{bos}\sigma}, J_b^{\text{bos}\sigma}] &= f_a^c{}_b J_c^{\text{bos}\sigma} \\ [J_{a^\sigma}^{\text{bos}\sigma}, J_{b^{-\sigma}}^{\text{bos}-\sigma}] &= 0 \end{aligned} \quad (2.69)$$

holds only for special k (in particular for the trivial case $k = 0$). In §3.1 (p.39) it is discussed how the k -vector field generating reparameterizations on loop space has this property and hence how the familiar $D = 2$ SWZW model is reobtained from the present approach.

SWZW backgrounds of course play an important role in practice because due to their high symmetry they allow exact solutions. They are therefore natural starting points for any perturbation theory in the background fields. Thus the typical perturbative calculation along the lines of §2.2.4 (p.31) and §2.2.5 (p.35) below will start with an exactly known spectrum on an SWZW background and then perturb $g_{\mu\nu}$ and $b_{\mu\nu}$ away from that. In the next section we develop the general Hamiltonian formalism needed for such calculations.

2.2 Covariant parameter evolution

2.2.1 Target space Killing evolution

We now set out to develop a machinery of parameter evolution obtained from supersymmetry constraints, which will be the basis of a covariant perturbation theory for systems described by such constraints.

Parameter evolution from the constraints. A generator of (target space) time evolution can be obtained from constraint equations of the form (2.1) if (\mathcal{M}, g) admits a timelike Killing vector v_0 : The observable associated with the “observer” v_0 is \mathcal{L}_{v_0} (and, in general, not $v_0^\mu \partial_\mu$ or $v_0^\mu \partial_\mu^c$ or the like), since this is invariantly defined and furthermore “gauge invariant” in the sense that it commutes with the constraints:

$$[\mathbf{D}_\pm, \mathcal{L}_{v_0}] = 0. \quad (2.70)$$

At this point we first assume that $\mathbf{D}_\pm = \mathbf{d} \pm \mathbf{d}^\dagger$ are the Dirac operators associated with the ordinary, undeformed, exterior derivative. The following construction will then be generalized step by step to the deformed cases.

Since $v_0 \cdot \hat{\gamma}_\pm$ is an invertible operator, the equivalences

$$\begin{aligned} \mathbf{D}_\pm \omega &= 0 \\ \Leftrightarrow v_0 \cdot \hat{\gamma}_\mp \mathbf{D}_\pm \omega &= 0 \\ \Leftrightarrow (v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- \pm v_0 \cdot \hat{\gamma}_- \mathbf{D}_+) \omega &= 0 \\ \Leftrightarrow (\{v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-\} \pm \{v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+\}) \omega &= -([v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-] \pm [v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+]) \omega \\ \stackrel{(A.80)(A.81)}{\Leftrightarrow} \begin{cases} 2(\partial_\mu v_{0\mu}) (\hat{c}^\dagger{}^\mu \hat{c}^\dagger{}^\nu + \hat{c}^\mu \hat{c}^\nu) \omega &= -([v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-] + [v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+]) \omega \\ 4\mathcal{L}_{v_0} \omega &= -([v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-] - [v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+]) \omega \end{cases} \end{aligned} \quad (2.71)$$

hold. The last line of (2.71) has the form of a Schrödinger equation

$$i\mathcal{L}_{v_0} \omega = \mathbf{H}_{v_0} \omega, \quad (2.72)$$

where the Hamiltonian \mathbf{H}_{v_0} is defined by

$$\begin{aligned} \mathbf{H}_{v_0} &:= \frac{i}{4} ([v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+] - [v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-]) \\ &= \frac{i}{2} ([v_0 \cdot \hat{c}^\dagger, \mathbf{d}^\dagger] - [v_0 \cdot \hat{c}, \mathbf{d}]) \\ &= \frac{i}{2} (v_0 \cdot \hat{\gamma}_- \mathbf{D}_+ - v_0 \cdot \hat{\gamma}_+ \mathbf{D}_-) + i\mathcal{L}_{v_0}. \end{aligned} \quad (2.73)$$

For special cases this Hamiltonian is indeed well known: For instance for flat Minkowski background it is the sum of two copies of the ordinary Hamiltonian of the Dirac electron:

$$\mathbf{H}_{v_0} = \frac{1}{2} (\hat{\gamma}_-^0 \hat{\gamma}_-^i - \hat{\gamma}_+^0 \hat{\gamma}_+^i) i\partial_i, \quad (\text{for } g_{\mu\nu} = \eta_{\mu\nu} \text{ and } v_0 = \partial_0). \quad (2.74)$$

In the context of classical electromagnetism in turn it is known as the Maxwell operator, which generates time evolution of the electromagnetic field. (This is discussed in more detail in C (p.73).) It is remarkable that a generalization of these well known Hamiltonians plays a role for supersymmetric quantum systems and indeed for the superstring. Heuristically this can be understood from the fact that both the Dirac particle as well as form field quanta appear in the massless sector of the superstring.

The Hamiltonian (2.73) is “time independent” in the sense that (by (A.82))

$$[\mathcal{L}_{v_0}, \mathbf{H}_{v_0}] = 0. \quad (2.75)$$

The left hand side of the second but last line in (2.71) gives a measure for how the v_0 -evolution of the $\hat{\gamma}_+$ sector deviates from that of the $\hat{\gamma}_-$ sector: If the curl $\partial_{[\mu}v_{0\nu]}$ of the Killing vector vanishes (which is equivalent to v_0 being covariantly constant), then $[v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-] = -[v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+]$ on states that satisfy the constraints, and the Hamiltonian reduces to

$$\mathbf{H}_{v_0} = \frac{i}{2} [v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+] = -\frac{i}{2} [v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-] \quad (\text{on-shell and for } \partial_{[\mu}v_{0\nu]} = 0) \quad (2.76)$$

Killing-deformed Hamiltonian. The generalization of all this to the Killing-deformed operators $\mathbf{D}_{k,\pm}$ (2.14) is straightforward: The analogue of (2.71) is

$$\begin{aligned} \mathbf{D}_{k,\pm} \omega &= 0 \\ \Leftrightarrow \begin{cases} \left(2(\partial_{[\mu}v_{0\nu]}) (\hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu} + \hat{c}^\mu \hat{c}^\nu) + 4iv_{0\mu} k^\mu \right) \omega &= - \left([v_0 \cdot \hat{\gamma}_+, \mathbf{D}_{k,-}] + [v_0 \cdot \hat{\gamma}_-, \mathbf{D}_{k,+}] \right) \omega \\ 4\mathcal{L}_{v_0} \omega &= - \left([v_0 \cdot \hat{\gamma}_+, \mathbf{D}_{k,-}] - [v_0 \cdot \hat{\gamma}_-, \mathbf{D}_{k,+}] \right) \omega \end{cases} \end{aligned} \quad (2.77)$$

Since the left hand side of the lower line remains unchanged, the deformed Hamiltonian \mathbf{H}_{k,v_0} is again of the form

$$\begin{aligned} \mathbf{H}_{k,v_0} &:= \frac{i}{4} \left([v_0 \cdot \hat{\gamma}_-, \mathbf{D}_{k,+}] - [v_0 \cdot \hat{\gamma}_+, \mathbf{D}_{k,-}] \right) \\ &\stackrel{(2.14)}{=} \mathbf{H}_{v_0} + \frac{1}{4} \left([v_0 \cdot \hat{\gamma}_-, k \cdot \hat{\gamma}_-] + [v_0 \cdot \hat{\gamma}_+, k \cdot \hat{\gamma}_+] \right). \end{aligned} \quad (2.78)$$

If \mathcal{L}_{v_0} is still to commute with the constraints, we need, due to (A.73), to require that

$$[v_0, k] \stackrel{!}{=} 0. \quad (2.79)$$

As shown in §3.1 (p.39) this condition indeed holds for the vector k used for representing the superstring on loop space.

In order that the Hamiltonian \mathbf{H} generates proper unitary evolution we need a scalar product on states (restricted to hypersurfaces perpendicular to the “time” direction induced by v_0) with respect to which \mathbf{H} is self-adjoint. This is the subject of the next section.

2.2.2 Scalar product.

One notable point about the Hamiltonian (2.73), or its generalization (2.78), is that it is *anti*-hermitian with respect to the Hodge inner product $\langle \cdot | \cdot \rangle$ (cf. (A.2)):

$$\mathbf{H}_{v_0}^\dagger = -\mathbf{H}_{v_0} . \quad (2.80)$$

For doing quantum mechanics we therefore need to construct, as in (A.21), a scalar product $\langle \cdot | \cdot \rangle_{\hat{\eta}}$ from the indefinite $\langle \cdot | \cdot \rangle$ with respect to which the Hamiltonian is a self-adjoint operator.

The obvious generalization of (A.20) is

$$\hat{\eta} := \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ . \quad (2.81)$$

This $\hat{\eta}$ obviously satisfies

$$\begin{aligned} \hat{\eta}^2 &= 1 \\ \hat{\eta}^\dagger &= \hat{\eta} \end{aligned} \quad (2.82)$$

and

$$[\mathcal{L}_{v_0}, \hat{\eta}] = 0 . \quad (2.83)$$

Because of

$$\hat{\eta} \mathbf{H}_{v_0} \hat{\eta} = -\mathbf{H}_{v_0} \quad (2.84)$$

the v_0 -Hamiltonian is indeed self-adjoint with respect to $\langle \cdot | \cdot \rangle_{\hat{\eta}}$:

$$\begin{aligned} \mathbf{H}_{v_0}^{\dagger \hat{\eta}} &= (\hat{\eta} \mathbf{H}_{v_0} \hat{\eta}^{-1})^\dagger \\ &= \hat{\eta} \mathbf{H}_{v_0}^\dagger \hat{\eta} \\ &= \mathbf{H}_{v_0} . \end{aligned} \quad (2.85)$$

To see that this makes sense, assume that target space \mathcal{M} is *static* and foliate \mathcal{M} by spacelike hypersurfaces orthogonal to the timelike Killing vector field v_0 . Let t_{v_0} be the coordinate that parametrizes the flow lines of v_0 , defined by

$$\mathbf{d} t_{v_0} = \frac{1}{v_0 \cdot v_0} v_{0\mu} dx^\mu , \quad (2.86)$$

so that

$$[\mathcal{L}_{v_0}, t_{v_0}] = 1 \quad (2.87)$$

and

$$[\mathbf{D}_\mp, t_{v_0}] = \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_\pm \quad (2.88)$$

Clearly, the Hamiltonian \mathbf{H}_{v_0} commutes with this time variable:

$$\begin{aligned} [\mathbf{H}_{v_0}, t_{v_0}] &= \left[\frac{i}{4} \left([v_0 \cdot \hat{\gamma}_-, \mathbf{D}_+] - [v_0 \cdot \hat{\gamma}_+, \mathbf{D}_-] \right), t_{v_0} \right] \\ &= \frac{1}{v_0 \cdot v_0} \frac{i}{4} \left([v_0 \cdot \hat{\gamma}_-, v_0 \cdot \hat{\gamma}_-] - [v_0 \cdot \hat{\gamma}_+, v_0 \cdot \hat{\gamma}_+] \right) \\ &= 0. \end{aligned} \quad (2.89)$$

The manifold \mathcal{M} is foliated into “equal time” slices by the mapping $t \mapsto \Sigma_{v_0}(t) := \{p \in \mathcal{M} | t_{v_0}(p) = t\}$, and the induced metric h on each leaf is

$$h_{\mu\nu} = g_{\mu\nu} - \frac{1}{v_0 \cdot v_0} v_{0\mu} v_{0\nu}. \quad (2.90)$$

The determinant of the full metric tensor then splits as

$$\sqrt{-g} = \sqrt{-v_0 \cdot v_0} \sqrt{h}. \quad (2.91)$$

Since \mathbf{H}_{v_0} generates unitary evolution from one Σ_{v_0} to the next, the scalar product of physical states should be restricted to a fixed (but arbitrary) hyperslice. Hence define

$$\begin{aligned} \langle \cdot | \cdot \rangle_{v_0} &:= \langle \cdot | \delta(t_{v_0}) | \cdot \rangle_{\hat{\eta}} \\ &= \langle \cdot | \delta(t_{v_0}) \hat{\eta} | \cdot \rangle. \end{aligned} \quad (2.92)$$

Because of (2.89) the Hamiltonian is still hermitian with respect to (2.92),

$$\mathbf{H}_{v_0}^{\dagger_{v_0}} = \mathbf{H}_{v_0}, \quad (2.93)$$

and hence generates a unitary evolution along t_{v_0} (*cf.* p. 26), implying in particular that

$$\mathcal{L}_{v_0} \langle \omega | \omega \rangle_{v_0} = 0 \quad (\text{for } \mathbf{D}_{\pm} \omega = 0). \quad (2.94)$$

It is easily checked that no problems arise when adapting this to the k -deformed case: The deformed Hamiltonian \mathbf{H}_{k,v_0} is also anti-hermitian with respect to $\langle \cdot | \cdot \rangle$ and hermitian with respect to $\langle \cdot | \cdot \rangle_{v_0}$:

$$(\mathbf{H}_{k,v_0})^{\dagger_{\hat{\eta}}} = \mathbf{H}_{k,v_0} \quad (2.95)$$

(The proof is given in B (p.69)).

Taking everything together, the proper scalar product reads explicitly (*cf.* (A.7))

$$\langle \alpha | \beta \rangle_{v_0} = \int_{\mathcal{M}, t_{v_0}=0} \langle \alpha | v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ | \beta \rangle_{\text{loc}} \frac{1}{\sqrt{-v_0 \cdot v_0}} \sqrt{h} d^{D-1}x. \quad (2.96)$$

This scalar product is related to the timelike component of a conserved current: For a given ω define the tensor $T^{\mu\nu}$ by

$$T^{\mu\nu} := \langle \omega | \hat{\gamma}^\mu_- \hat{\gamma}^\nu_+ | \omega \rangle_{\text{loc}} . \quad (2.97)$$

If ω satisfies the constraint $\mathbf{D}_\pm \omega = 0$ or $\mathbf{D}_{k,\pm} \omega = 0$, then this tensor is conserved,

$$\nabla_\mu T^{\mu\nu} = 0 \quad (2.98)$$

and in particular

$$P^\mu_{v_0} := T^\mu_{\nu} v_0^\nu \quad (2.99)$$

is a conserved current and

$$\begin{aligned} \langle \omega | \omega \rangle_{v_0} &= \int_{\mathcal{M}, t_{v_0}=0} \langle \omega | v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ | \omega \rangle_{\text{loc}} \sqrt{h} d^{D-1}x \\ &= \int_{\mathcal{M}, t_{v_0}=0} \frac{1}{\sqrt{-v_0 \cdot v_0}} v_0 \cdot P_{v_0} \sqrt{h} d^{D-1}x . \end{aligned} \quad (2.100)$$

In coordinates adapted to the foliation we have

$$\begin{aligned} v_0^\mu &= \delta_0^\mu \\ v_{0\mu} &= g_{\mu 0} = \delta_\mu^0 v_0 \cdot v_0 \end{aligned} \quad (2.101)$$

and hence the scalar product is time independent:

$$\begin{aligned} \mathcal{L}_{v_0} \langle \omega | \omega \rangle_{v_0} &= \partial_0 \int_{\mathcal{M}, t_{v_0}=0} P_{v_0}^0 \sqrt{-v_0 \cdot v_0} \sqrt{h} d^{D-1}x \\ &= \int_{\mathcal{M}, t_{v_0}=0} \partial_0 \left(\sqrt{-g} P_{v_0}^0 \right) d^{D-1}x \\ &= - \int_{\mathcal{M}, t_{v_0}=0} \partial_i \left(\sqrt{-g} P_{v_0}^i \right) d^{D-1}x \\ &= 0 . \end{aligned} \quad (2.102)$$

When restricted to the undeformed case and to 2-form fields these constructions reduce to relations well known from Dirac theory and classical electromagnetism, see (C.10).

The formalism so far gives us a unitary Hamiltonian evolution along a timelike Killing vector field obtained from the supersymmetry constraints $\mathbf{D}_\pm \omega = 0$. But according to (2.71) the associated Schrödinger equation contains only part of the physical content of these constraints. Further information is contained in the second but last line of (2.71). We will now show that this yields a constraint on states restricted to spacelike hypersurfaces which is compatible with our Hamiltonian \mathbf{H} .

Propagator and physical states. It is convenient to introduce the further abbreviation

$$\mathbf{C}_{v_0} := v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- + v_0 \cdot \hat{\gamma}_- \mathbf{D}_+, \quad (2.103)$$

so that (2.1) is equivalently rewritten as

$$\mathbf{D}_\pm \omega = 0 \Leftrightarrow \begin{cases} i\mathcal{L}_{v_0} \omega = \mathbf{H}_{v_0} \omega \\ \mathbf{C}_{v_0} \omega = 0 \end{cases}. \quad (2.104)$$

Because of

$$\begin{aligned} [t_{v_0}, \mathbf{C}_{v_0}] &= 0 \\ [\mathcal{L}_{v_0}, \mathbf{C}_{v_0}] &= 0 \end{aligned} \quad (2.105)$$

the constraint \mathbf{C}_{v_0} must hold on each hyperslice Σ_t separately:

$$\begin{aligned} \mathbf{C}_{v_0} \omega &= 0 \\ \Rightarrow \mathbf{C}_{v_0} (\delta(t_{v_0}, t) \omega) &= 0. \end{aligned} \quad (2.106)$$

In fact, every form ω_0 on Σ_0 , which satisfies the spatial constraint $\mathbf{C}_{v_0} \omega_0 = 0$, uniquely corresponds to a state ω on all of \mathcal{M} , given by

$$\omega = \exp(-i\mathbf{H}t_{v_0}) \omega_0, \quad (2.107)$$

that satisfies the full constraints (2.104). In order to see this, note that the constraint \mathbf{C}_{v_0} *commutes weakly* with the Hamiltonian \mathbf{H}_{v_0} , i.e. up to a term that vanishes when the spatial constraints are fulfilled:

$$[\mathbf{C}_{v_0}, \mathbf{H}_{v_0}] = \frac{1}{2iv_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ \mathbf{C}_{v_0}. \quad (2.108)$$

(The proof of this is given in appendix B (p.71).) Hence we have

$$\begin{aligned} \mathbf{C}_{v_0} (1 - i\mathbf{H}_{v_0} \epsilon) \omega_0 &= i\epsilon [\mathbf{H}_{v_0}, \mathbf{C}_{v_0}] \omega_0 \\ &\stackrel{(2.108)}{=} 0, \quad (\text{for } \mathbf{C}_{v_0} \omega_0 = 0) \end{aligned} \quad (2.109)$$

for any constant ϵ . Iterating this argument yields

$$\mathbf{C}_{v_0} (1 - i\mathbf{H}_{v_0} t_{v_0}/n)^n \omega_0 = 0, \quad \text{for } n \in \mathbb{N} \text{ and } \mathbf{C}_{v_0} \omega_0 = 0, \quad (2.110)$$

which in the limit $n \rightarrow \infty$ gives

$$\mathbf{C}_{v_0} \exp(-i\mathbf{H}_{v_0} t_{v_0}) \omega_0 = 0, \quad (\text{for } \mathbf{C}_{v_0} \omega_0 = 0). \quad (2.111)$$

Since, by assumption, $\mathcal{L}_{v_0}\omega_0 = 0$, the state ω of course also satisfies the Schrödinger equation:

$$i\mathcal{L}_{v_0}\exp(-i\mathbf{H}_{v_0}t_{v_0})\omega_0 = \mathbf{H}_{v_0}\exp(-i\mathbf{H}_{v_0}t_{v_0})\omega_0. \quad (2.112)$$

The generalization to the k -deformed case is again unproblematic, since one finds in perfect analogy with (2.108) that

$$[\mathbf{C}_{k,v_0}, \mathbf{H}_{k,v_0}] = \frac{1}{2iv_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ \mathbf{C}_{k,v_0}. \quad (2.113)$$

(The proof is given in appendix B on p. 72.)

The above constructions show that a covariant Hamiltonian with all the familiar properties can be constructed in $D = 1$ and $D = 2$ supersymmetric systems with purely gravitational background. We now want to generalize all this to the case where there is additionally a non-vanishing b -field background. In order to do so we make use of the fact that the supersymmetry constraints in such backgrounds are obtained from those of the already understood backgrounds by a deformation induced by the deformation operator (2.46).

2.2.3 Parameter evolution in the presence of a B -field

The parameter evolution that we are interested in requires that the background fields be “time independent”. Hence all background fields must have vanishing Lie derivative along v_0 . For the b -field this is equivalent to (*cf.* (2.46))

$$[\mathcal{L}_{v_0}, \mathbf{W}^{(b)}] = 0. \quad (2.114)$$

Recall (2.47) that the b -field induces on \mathbf{d} and \mathbf{d}^\dagger the deformation

$$\begin{aligned} \mathbf{d}^{(b)} &:= e^{-\mathbf{W}^{(b)}} \mathbf{d} e^{\mathbf{W}^{(b)}} \\ \mathbf{d}^{\dagger(b)} &:= e^{\mathbf{W}^{\dagger(b)}} \mathbf{d}^\dagger e^{-\mathbf{W}^{\dagger(b)}}. \end{aligned} \quad (2.115)$$

Since for further constructions it will be essential to have an analogue of (A.69) and (A.79) and hence of (2.71), we define the following deformations of the form creators and annihilators:

$$\begin{aligned} \hat{\mathbf{c}}^{\dagger(b)\mu} &:= e^{\mathbf{W}^\dagger} \hat{\mathbf{c}}^{\dagger\mu} e^{-\mathbf{W}^\dagger} \\ &= \hat{\mathbf{c}}^{\dagger\mu} + \left[\hat{\mathbf{c}}^{\dagger\mu}, \frac{1}{2} b_{\alpha\beta} \hat{\mathbf{c}}^\alpha \hat{\mathbf{c}}^\beta \right] \\ &= \hat{\mathbf{c}}^{\dagger\mu} + b^\mu{}_\beta \hat{\mathbf{c}}^\beta \\ \hat{\mathbf{c}}^{(b)\mu} &:= e^{-\mathbf{W}} \hat{\mathbf{c}}^\mu e^{\mathbf{W}} \\ &= \hat{\mathbf{c}}^\mu + \left[\hat{\mathbf{c}}^\mu, \frac{1}{2} b_{\alpha\beta} \hat{\mathbf{c}}^{\dagger\alpha} \hat{\mathbf{c}}^{\dagger\beta} \right] \\ &= \hat{\mathbf{c}}^\mu + b^\mu{}_\beta \hat{\mathbf{c}}^{\dagger\beta}. \end{aligned} \quad (2.116)$$

The purpose of this definition is that now the relations

$$\begin{aligned}
\{v_0 \cdot \hat{c}^{(b)}, \mathbf{d}^{(b)}\} &= e^{-\mathbf{W}^{(b)}} \{v_0 \cdot \hat{c}, \mathbf{d}\} e^{\mathbf{W}^{(b)}} \\
&= e^{-\mathbf{W}^{(b)}} \mathcal{L}_{v_0} e^{\mathbf{W}^{(b)}} \\
&\stackrel{(2.114)}{=} \mathcal{L}_{v_0}
\end{aligned} \tag{2.117}$$

and

$$\{v_0 \cdot \hat{c}^{\dagger(b)}, \mathbf{d}^{\dagger(b)}\} = -\mathcal{L}_{v_0} \tag{2.118}$$

hold, and analogously for the k -deformed case:

$$\begin{aligned}
\{v_0 \cdot \hat{c}^{(b)}, \mathbf{d}_k^{(b)}\} &= e^{-\mathbf{W}^{(b)}} (\{v_0 \cdot \hat{c}, \mathbf{d}\} + \{v_0 \cdot \hat{c}, ik \cdot \hat{c}\}) e^{\mathbf{W}^{(b)}} \\
&= \mathcal{L}_{v_0} \\
\{v_0 \cdot \hat{c}^{\dagger(b)}, \mathbf{d}_k^{\dagger(b)}\} &= -\mathcal{L}_{v_0} .
\end{aligned} \tag{2.119}$$

The deformed creators and annihilators satisfy

$$\begin{aligned}
\{\hat{c}^{\dagger(b)\mu}, \hat{c}^{\dagger(b)\nu}\} &= 0 \\
\{\hat{c}^{(b)\mu}, \hat{c}^{(b)\nu}\} &= 0 \\
\{\hat{c}^{(b)\mu}, \hat{c}^{\dagger(b)\nu}\} &= g^{\mu\nu} + b^\mu{}_\alpha b^{\nu\alpha} \\
&= g^{(b)\mu\nu} .
\end{aligned} \tag{2.120}$$

The tensor

$$g_{\mu\nu}^{(b)} := (g - bg^{-1}b)_{\mu\nu} \tag{2.121}$$

is known in string theory as the *open string metric* in the presence of a b -field (*cf.* [24], p.9). Note that even though it plays a role similar to a metric tensor, we will never shift indices with anything but the ordinary metric g . In particular

$$g^{(b)\mu\nu} := (g + bg^{-1}b)_{\mu'\nu'} g^{\mu'\mu} g^{\nu'\nu} . \tag{2.122}$$

The b -deformed analogue of the Clifford generators (A.4) is of course

$$\begin{aligned}
\hat{\gamma}_\pm^{(b)\mu} &:= \hat{c}^{\dagger(b)\mu} \pm \hat{c}^{(b)\mu} \\
&= \hat{\gamma}_\pm^\mu \pm b^\mu{}_\alpha \hat{\gamma}_\pm^\alpha \\
&= (e_a{}^\mu \mp b_a{}^\mu) \hat{\gamma}_\pm^\alpha ,
\end{aligned} \tag{2.123}$$

satisfying

$$\begin{aligned}
\{\hat{\gamma}_\pm^{(b)\mu}, \hat{\gamma}_\mp^{(b)\nu}\} &= 0 \\
\{\hat{\gamma}_\pm^{(b)\mu}, \hat{\gamma}_\pm^{(b)\nu}\} &= \pm 2g^{(b)\mu\nu} .
\end{aligned} \tag{2.124}$$

We will often need the covariant version of the deformed Clifford generators (2.123):

$$\hat{\gamma}_{\mu\pm}^{(b)} = (e_\mu^a \pm b_\mu^a) \hat{\gamma}_{a\pm} . \quad (2.125)$$

Equations (2.123) and (2.125) motivate the introduction of the b -deformed version of the vielbein e_a^μ and its inverse e_μ^a :

$$\begin{aligned} e_{\pm a}^{(b)\mu} &:= e_a^\mu \mp b_a^\mu \\ e_{\pm \mu}^{(b)a} &:= e_\mu^a \pm b_\mu^a . \end{aligned} \quad (2.126)$$

In terms of these we can write succinctly

$$\begin{aligned} \hat{\gamma}_\pm^{(b)} &= e_{\pm a}^{(b)\mu} \hat{\gamma}_\pm^a \\ \hat{\gamma}_{\mu\pm}^{(b)} &= e_{\pm \mu}^{(b)a} \hat{\gamma}_{a\pm} \end{aligned} \quad (2.127)$$

The purpose of all this is that using (2.119) it is now immediate that, in complete analogy with (2.71), we have

$$\{v_0 \cdot \hat{\gamma}_+^{(b)}, \mathbf{D}_-^{(b)}\} - \{v_0 \cdot \hat{\gamma}_-^{(b)}, \mathbf{D}_+^{(b)}\} = 4\mathcal{L}_{v_0} . \quad (2.128)$$

This means that the construction (2.73) of a Hamiltonian generator of parameter evolution carries over to the b -deformed case as follows:

$$\begin{aligned} \mathbf{H}_{v_0}^{(b)} &= \frac{i}{4} \left([v_0 \cdot \hat{\gamma}_+^{(b)}, \mathbf{D}_-^{(b)}] - [v_0 \cdot \hat{\gamma}_-^{(b)}, \mathbf{D}_+^{(b)}] \right) \\ &= \frac{i}{2} \left(\hat{\gamma}_-^{(b)} \mathbf{D}_+^{(b)} - \hat{\gamma}_+^{(b)} \mathbf{D}_-^{(b)} \right) + i\mathcal{L}_{v_0} . \end{aligned} \quad (2.129)$$

Noting that (by (2.116) and (2.114))

$$\left[\mathcal{L}_{v_0}, v_0 \cdot \hat{\mathbf{c}}^{\dagger(b)} \right] = 0 = \left[\mathcal{L}_{v_0}, v_0 \cdot \hat{\mathbf{c}}^{(b)} \right] , \quad (2.130)$$

it is easy to see (the details are given in appendix B (p.69)) that this Hamiltonian is self-adjoint with respect to the scalar product induced by the appropriately deformed Krein space operator (*cf.* (2.81))

$$\hat{\eta}^{(b)} := \left(v_0 \cdot \hat{\gamma}_-^{(b)} \right)^{-2} v_0 \cdot \hat{\gamma}_-^{(b)} v_0 \cdot \hat{\gamma}_+^{(b)} , \quad (2.131)$$

i.e.

$$\mathbf{H}_{v_0}^{\dagger \hat{\eta}^{(b)}} = \mathbf{H}_{v_0} . \quad (2.132)$$

But care has to be exercised, since $\hat{\eta}$ -hermiticity is not sufficient for many applications. What really matters is hermiticity with respect to the time-reparameterization gauge fixed scalar product (2.92) induced by

$$\hat{\eta}_{v_0}^{(b)} = \hat{\eta}^{(b)} \delta(t_{v_0}) . \quad (2.133)$$

An $\hat{\eta}$ -hermitian operator $A = A^{\dagger_{\hat{\eta}}}$ is v_0 -hermitian if it commutes with the time coordinate t_{v_0} defined by (2.86). This is not the case for the operator (2.129) (*cf.* (2.89)):

$$[\mathbf{H}_{v_0}^{(b)}, t_{v_0}] = \frac{1}{v_0 \cdot v_0} \frac{i}{4} \left([v_0 \cdot \hat{\gamma}_+^{(b)}, v_0 \cdot \hat{\gamma}_+] - [v_0 \cdot \hat{\gamma}_-^{(b)}, v_0 \cdot \hat{\gamma}_-] \right). \quad (2.134)$$

This failure to be v_0 -hermitian can be remedied by adding an appropriate correction operator. Define

$$\tilde{\mathbf{H}}_{v_0}^{(b)} := \mathbf{H}_{v_0}^{(b)} - \frac{1}{v_0 \cdot v_0} \frac{i}{4} \left([v_0 \cdot \hat{\gamma}_+^{(b)}, v_0 \cdot \hat{\gamma}_+] - [v_0 \cdot \hat{\gamma}_-^{(b)}, v_0 \cdot \hat{\gamma}_-] \right) \mathcal{L}_{v_0}. \quad (2.135)$$

This operator is $\hat{\eta}_{v_0}^{(b)}$ -hermitian (by the same argument as in (B.9)) and by construction commutes with t_{v_0} , therefore it is $\hat{\eta}_{v_0}^{(b)}$ -hermitian:

$$(\tilde{\mathbf{H}}_{v_0}^{(b)})^{\dagger_{\hat{\eta}_{v_0}^{(b)}}} = \tilde{\mathbf{H}}_{v_0}^{(b)}. \quad (2.136)$$

On physical states $|\phi\rangle$ this operator satisfies

$$\left(1 - \frac{1}{v_0 \cdot v_0} \frac{1}{4} \left([v_0 \cdot \hat{\gamma}_+^{(b)}, v_0 \cdot \hat{\gamma}_+] - [v_0 \cdot \hat{\gamma}_-^{(b)}, v_0 \cdot \hat{\gamma}_-] \right) \right) i \mathcal{L}_{v_0} |\phi\rangle = \tilde{\mathbf{H}}_{v_0}^{(b)} |\phi\rangle. \quad (2.137)$$

We write

$$\begin{aligned} K &:= \frac{1}{v_0 \cdot v_0} \frac{1}{4} \left([v_0 \cdot \hat{\gamma}_+^{(b)}, v_0 \cdot \hat{\gamma}_+] - [v_0 \cdot \hat{\gamma}_-^{(b)}, v_0 \cdot \hat{\gamma}_-] \right) \\ &= \frac{1}{2v_0 \cdot v_0} \left(v_0 \cdot \hat{\gamma}_+^{(b)} v_0 \cdot \hat{\gamma}_+ - v_0 \cdot \hat{\gamma}_-^{(b)} v_0 \cdot \hat{\gamma}_- \right) - 1 \\ &= v_0^\mu b_{\mu\nu} \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ + v_0^\mu b_{\mu\nu} \hat{\gamma}_-^\nu v_0 \cdot \hat{\gamma}_- \end{aligned} \quad (2.138)$$

for the operator on the left. This operator is $\hat{\eta}_{v_0}^{(b)}$ -hermitian

$$K^{\dagger_{\hat{\eta}_{v_0}^{(b)}}} = K. \quad (2.139)$$

In terms of K equation (2.137) becomes

$$(1 - K) i \mathcal{L}_{v_0} |\phi\rangle = \tilde{\mathbf{H}}_{v_0}^{(b)} |\phi\rangle. \quad (2.140)$$

This is the modified form of the Schrödinger equation that needs to be used whenever it is crucial that the operator on the right hand side really is self-adjoint with respect to $\langle \cdot | \hat{\eta}_{v_0}^{(b)} | \cdot \rangle$, with $\hat{\eta}_{v_0}^{(b)}$ given by (2.133), i.e. that it really commutes with the evolution parameter t_{v_0} . This will in particular be necessary in perturbation theory (see §2.2.5 (p.35)).

Parameter evolution in the presence of torsion. Often in the literature a B -field background is addressed as a torsion background. This is justified since, as discussed in §2.1.3 (p.15) (*cf.* (2.49)), the B -field induces a deformation of the covariant derivative which makes it act like the covariant derivative with torsion $\propto +dB$ on one spinor bundle and with torsion $\propto -dB$ on the other. However, this deformed operator is of course not the covariant derivative on the exterior bundle that one would ordinarily associate with a connection of non-vanishing torsion. Instead, the latter is, as discussed in §A.5 (p.65), given by expression (A.99).

The deformation of the supersymmetry generators associated with (A.99), which one might perhaps naively associate with a “torsion background”, does not arise in string theory. Nevertheless, because it is interesting in itself, we mention that for this case, too, one can carry out the program of §2.2.1:

So consider replacing the constraints (2.1) by their torsion-deformed versions (A.106):

$$\mathbf{D}_{T,\pm}\omega = 0, \quad (2.141)$$

for some non-vanishing antisymmetric torsion tensor $T_{\mu\alpha\beta}$. Then the construction (2.71) gives rise to the torsion-deformed Lie derivative operators

$$\begin{aligned} \mathcal{L}_{T,v} &:= \{\mathbf{d}_T, v_\mu \hat{c}^\mu\} \\ &= \{\mathbf{d}, v^\mu \hat{c}_\mu\} - \left\{ T_\mu{}^\alpha{}_\beta \hat{c}^{\dagger\mu} \hat{c}^{\dagger\beta} \hat{c}_\alpha, v^\mu \hat{c}_\mu \right\} \\ &= \mathcal{L}_v - 2v^\mu T_\mu{}^\alpha{}_\beta \hat{c}^{\dagger\beta} \hat{c}_\alpha. \end{aligned} \quad (2.142)$$

Because the term on the right is anti-hermitian for all v , the operator $\mathcal{L}_{T,v}$ still satisfies the crucial condition (A.76):

$$(\mathcal{L}_{T,v})^\dagger = -\mathcal{L}_{T,v} \Leftrightarrow v \text{ Killing}. \quad (2.143)$$

Furthermore the covariant derivative of a Killing vector v_μ with respect to ω_T is still antisymmetric:

$$\nabla_{T,\mu} v_\nu = \nabla_{T,[\mu} v_{\nu]} \quad (v \text{ Killing}). \quad (2.144)$$

This is the condition that the proofs B (p.71) rely on. Hence they carry over to the torsion deformed case and we can straightforwardly generalize (2.72), (2.73) and (2.108) to the case of non-vanishing torsion by replacing \mathcal{L}_{v_0} by \mathcal{L}_{T,v_0} and \mathbf{D}_\pm by $\mathbf{D}_{T,\pm}$ throughout.

2.2.4 Perturbation of background fields

We have now succeeded in constructing covariant Hamiltonian operators for general metric and Kalb-Ramond field backgrounds. Any perturbation of these background

fields will induce a perturbation of this Hamiltonian operator. Since there are some subtleties involved in calculating that perturbed Hamiltonian from the perturbed background fields we explicitly spell out the necessary steps in this section. The next section then shows how, given the perturbed Hamiltonian, the first order perturbation theory of ordinary quantum mechanics can be adapted to Schrödinger equations of the form (2.140).

A perturbation of the background fields labelled by a perturbation parameter ϵ

$$\begin{aligned} g_{\mu\nu} &\rightarrow \underbrace{g_{\mu\nu}^{(0)}}_{=g_{\mu\nu}} + \sum_{n=1}^{\infty} \underbrace{g_{\mu\nu}^{(n)}}_{\mathcal{O}(\epsilon^n)} \\ b_{\mu\nu} &\rightarrow \underbrace{b_{\mu\nu}^{(0)}}_{=b_{\mu\nu}} + \sum_{n=1}^{\infty} \underbrace{b_{\mu\nu}^{(n)}}_{\mathcal{O}(\epsilon^n)} \end{aligned} \quad (2.145)$$

induces a deformation of the various operators considered here. This section briefly collects some of the relevant formulae, which will be needed in §2.2.5 (p.35) for writing down an expression for the first order energy shift.

In the σ -model Lagrangian the background fields act as coupling constants for the canonical fields $\Gamma_{\pm}^a, X^{\mu}, \partial_{X^{\mu}}$, which themselves therefore receive no perturbation:

$$\begin{aligned} \Gamma_{\pm}^a &\rightarrow \hat{\gamma}_{\pm}^a \\ X^{\mu} &\rightarrow X^{\mu} \\ \partial_{X^{\mu}} &\rightarrow \partial_{X^{\mu}}. \end{aligned} \quad (2.146)$$

Geometrically this means that while perturbing g and b the coordinates on the configuration manifold are fixed, as is the chosen ONB section of the two Clifford bundles:

$$\begin{aligned} [\partial_{X^{\mu}}, X^{\nu}] &= \delta_{\mu}^{\nu} \\ \{\Gamma_{a\pm}, \Gamma_{\pm}^b\} &= \pm 2\delta_a^b. \end{aligned} \quad (2.147)$$

The perturbed geometry is felt by the canonical fields via the perturbation of the vielbein

$$e_a^{\mu} = e_a^{(0)\mu} + \underbrace{e_a^{(1)\mu}}_{=\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^2). \quad (2.148)$$

This has to satisfy

$$\begin{aligned} ds^2(e_a, e_b) &\stackrel{!}{=} \eta_{ab} \\ &= \underbrace{ds_0^2(e_a^{(0)}, e_b^{(0)})}_{=\eta_{ab}} \\ &\quad + \underbrace{ds_1^2(e_a^{(0)}, e_b^{(0)}) + ds_0^2(e_a^{(0)}, e_b^{(1)}) + ds_0^2(e_b^{(0)}, e_a^{(1)})}_{\mathcal{O}(\epsilon) \stackrel{!}{=} 0} \end{aligned}$$

$$\begin{aligned}
& +ds_2^2(e_a^{(0)}, e_b^{(0)}) + ds_1^2(e_a^{(1)}, e_b^{(0)}) + ds_1^2(e_a^{(0)}, e_b^{(1)}) \\
& + \underbrace{ds_0^2(e_a^{(1)}, e_b^{(1)}) + ds_0^2(e_a^{(2)}, e_b^{(0)}) + ds_0^2(e_a^{(0)}, e_b^{(2)})}_{\mathcal{O}(\epsilon^2) \stackrel{!}{=} 0} \\
& + \mathcal{O}(\epsilon^3) ,
\end{aligned} \tag{2.149}$$

where we write $ds_n^2(v, w)$ for $g_{\mu\nu}^{(n)} v^\mu w^\nu$. Hence the first and second order perturbation of the vielbein is given by

$$\begin{aligned}
e_a^{(1)} &= \left(q_{ab}^{(1)} - \frac{1}{2} ds_{(1)}^2(e_a^{(0)}, e_b^{(0)}) \right) (e^{b(0)}) \\
e_a^{(2)} &= \left(q_{ab}^{(2)} - \frac{1}{2} \left(ds_{(2)}^2(e_a^{(0)}, e_b^{(0)}) + ds_1^2(e_a^{(1)}, e_b^{(0)}) + ds_1^2(e_a^{(0)}, e_b^{(1)}) + ds_0^2(e_a^{(1)}, e_b^{(1)}) \right) \right) (e^{b(0)}) ,
\end{aligned} \tag{2.150}$$

where

$$q_{ab} = -q_{ba} \tag{2.151}$$

is an arbitrary antisymmetric tensor which incorporates the gauge freedom in the choice of vielbein. The inverse vielbein is then

$$\begin{aligned}
e^a{}_\mu &= e_b{}^\nu \eta^{ab} g_{\mu\nu} \\
&= \underbrace{e_b^{(0)\nu} \eta^{ab} g_{\mu\nu}^{(0)}}_{\mathcal{O}(1)} + \underbrace{e_b^{(0)\nu} \eta^{ab} g_{\mu\nu}^{(1)} + e_b^{(1)\nu} \eta^{ab} g_{\mu\nu}^{(0)}}_{\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^2) .
\end{aligned} \tag{2.152}$$

For the “structure functions” $f_a{}^c{}_b$ of the vielbein one has

$$\begin{aligned}
[e_a, e_b] &= f_a{}^c{}_b e_c \\
\Rightarrow \underbrace{[e_a^{(0)}, e_b^{(0)}]}_{\mathcal{O}(\epsilon^0)} + \underbrace{[e_a^{(0)}, e_b^{(1)}] + [e_a^{(1)}, e_b^{(0)}]}_{\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^2) &= \underbrace{f_{a{}^c{}_b}^{(0)} e_c^{(0)}}_{\mathcal{O}(\epsilon^0)} + \underbrace{f_{a{}^c{}_b}^{(1)} e_c^{(0)} + f_{a{}^c{}_b}^{(0)} e_c^{(1)}}_{\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{2.153}$$

and therefore their first-order perturbation is found to be

$$\Rightarrow f_{a{}^c{}_b}^{(1)} = \left([e_a^{(0)}, e_b^{(1)}] + [e_a^{(1)}, e_b^{(0)}] - f_{a{}^{c'}{}_b}^{(0)} e_{c'}^{(1)} \right) \cdot e^{c(0)} . \tag{2.154}$$

Now the shift in the ONB connection

$$\omega_{abc} = \frac{1}{2} (f_{abc} + f_{bca} - f_{cab}) \tag{2.155}$$

is immediate:

$$\omega_{abc}^{(1)} = \frac{1}{2} (f_{abc}^{(1)} + f_{bca}^{(1)} - f_{cab}^{(1)}) . \tag{2.156}$$

The ONB components of the field strength

$$h_{abc} = e_a^\mu e_\nu^\mu e_a^\rho h_{\mu\nu\rho} \quad (2.157)$$

obviously receive the correction

$$h_{abc}^{(1)} = e_a^{(0)\mu} e_b^{(0)\nu} e_c^{(0)\rho} h_{\mu\nu\rho}^{(1)} + e_a^{(1)\mu} e_b^{(0)\nu} e_c^{(0)\rho} h_{\mu\nu\rho}^{(0)} + e_a^{(0)\mu} e_b^{(1)\nu} e_c^{(0)\rho} h_{\mu\nu\rho}^{(0)} + e_a^{(0)\mu} e_b^{(0)\nu} e_c^{(1)\rho} h_{\mu\nu\rho}^{(0)}. \quad (2.158)$$

Also the perturbation of the b -deformed covariant derivative operator (2.49) simply reads

$$\hat{\nabla}_a^{(b)(1)} := \frac{1}{4} \omega_{abc}^{+(1)} \hat{\gamma}^{b+} \hat{\gamma}^{c+} - \frac{1}{4} \omega_{abc}^{-(1)} \hat{\gamma}^{b-} \hat{\gamma}^{c-}, \quad (2.159)$$

where of course

$$\omega^{\pm(1)} := \omega^{(1)} \pm \frac{1}{2} h^{(1)}. \quad (2.160)$$

With these ingredients the perturbation of the supercharges (the Dirac operators) are found to be (*cf.* (2.48))

$$\mathbf{D}_{k^\mp}^{(b)(1)} = \hat{\gamma}_\pm^a \left(\hat{\nabla}_a^{(b)(1)} - i(b_{a\nu}^{(1)} \mp g_{a\nu}^{(1)}) k^\nu \right) - \frac{1}{12} h_{abc}^{(1)} \hat{\gamma}_\pm^a \hat{\gamma}_\pm^b \hat{\gamma}_\pm^c. \quad (2.161)$$

Here it is assumed that k^μ remains unperturbed, which is the case for the superstring, where $k^\mu \rightarrow X'^\mu(\sigma)$.

Finally this allows to write down an expression for the perturbation of the target-space Hamiltonian (2.129): In addition to the modification of the supercharges (2.161) the deformed Clifford generators (2.123) will receive a correction:

$$\begin{aligned} v_0 \cdot \hat{\gamma}_\pm^{(b)} &= v_{0,\mu} (e_a^\mu \pm b_a^\mu) \hat{\gamma}_\pm^a \\ &= [v_{0,\mu} (e_a^\mu \pm b_a^\mu)]^{(0)} \hat{\gamma}_\pm^a + [v_{0,\mu} (e_a^\mu \pm b_a^\mu)]^{(1)} \hat{\gamma}_\pm^a + \dots \\ &:= [v_0 \cdot \hat{\gamma}_\pm^{(b)}]^{(0)} + [v_0 \cdot \hat{\gamma}_\pm^{(b)}]^{(1)} + \dots \end{aligned} \quad (2.162)$$

Therefore there are two contributions to the perturbation of the Hamiltonian (2.129):

$$\begin{aligned} [\mathbf{H}_{v_0}^{(b)}]^{(1)} &= \frac{i}{4} \left([[v_0 \cdot \hat{\gamma}_+^{(b)}]^{(1)}, \mathbf{D}_-^{(b)}] - [[v_0 \cdot \hat{\gamma}_-^{(b)}]^{(1)}, \mathbf{D}_+^{(b)}] \right) \\ &\quad + \frac{i}{4} \left([v_0 \cdot \hat{\gamma}_+^{(b)}, [\mathbf{D}_-^{(b)}]^{(1)}] - [v_0 \cdot \hat{\gamma}_-^{(b)}, [\mathbf{D}_+^{(b)}]^{(1)}] \right). \end{aligned} \quad (2.163)$$

2.2.5 Perturbation theory

With a Hamiltonian generator of target space time evolution in hand, the standard techniques of quantum mechanical perturbation theory can be adapted. The differences that one has to deal with are the need for the Krein space operator $\hat{\eta}_{v_0}^{(b)}$ (2.133) and the presence of non-vanishing K in the modified Schrödinger equation (2.140), which may (but need not) appear in the presence of non-vanishing Kalb-Ramond backgrounds.

So what we are interested in is finding approximate solutions to the Eigenvalue problem

$$\begin{aligned} & \left[(1 - K) i\mathcal{L}_{v_0} - \tilde{\mathbf{H}}_{v_0}^{(b)} \right] |\phi\rangle = 0 \\ \Leftrightarrow & \left[(1 - K) E_n - \tilde{\mathbf{H}}_{v_0}^{(b)} \right] |\phi_n\rangle = 0 \end{aligned} \quad (2.164)$$

on the basis that a solution to 0th order in the perturbation is known

$$\left[(1 - K^{(0)}) E_n^{(0)} - \tilde{\mathbf{H}}_{v_0}^{(b)(0)} \right] |\phi_n^{(0)}\rangle = 0. \quad (2.165)$$

Because $\tilde{\mathbf{H}}_{v_0}^{(b)(0)}$ is hermitian with respect to $(\hat{\eta}_{v_0}^{(b)})^{(0)}$ it follows that the $\phi_n^{(0)}$ for different $E_n^{(0)}$ are orthogonal with respect to $\langle \cdot | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) | \cdot \rangle$. We shall assume that they form a complete basis. The completeness relation can then be written in the form

$$|\phi^{(0)}\rangle = \sum_n \frac{\langle \phi_n^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) | \phi^{(0)} \rangle}{\langle \phi_n^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) | \phi_n^{(0)} \rangle} |\phi_n^{(0)}\rangle. \quad (2.166)$$

In order to find an expression for the first order perturbation of eigenvalues and states we multiply equation (2.164) from the left by $\langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)}$, which gives

$$\langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} \left[(1 - (K^{(0)} + K^{(1)})) E_n - (\tilde{\mathbf{H}}_{v_0}^{(b)})^{(0)} - (\tilde{\mathbf{H}}_{v_0}^{(b)})^{(1)} \right] |\phi_n^{(0)} + \phi_n^{(1)}\rangle = 0 + \dots \quad (2.167)$$

up to terms of higher than first order. (Here $A^{(m)}$ is the mth order perturbation of the object A .) The point of taking the scalar product with respect to the *unperturbed* operator $\hat{\eta}_{v_0}^{(b)(0)}$ (see (2.133)) is that it allows us to use the hermiticity (2.139) of $(\tilde{\mathbf{H}}_{v_0}^{(b)})^{(0)}$ with respect to this scalar product to apply it to the left and write

$$\langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (\tilde{\mathbf{H}}_{v_0}^{(b)})^{(0)} \stackrel{(2.165)}{=} \langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) E_m^{(0)}. \quad (2.168)$$

The remaining occurrence of \mathcal{L}_{v_0} in $(\tilde{\mathbf{H}}_{v_0}^{(b)})^{(1)}$ can be applied to the right to give, as usual,

$$i\mathcal{L}_{v_0} |\phi_n\rangle = E_n |\phi\rangle = (E_n^{(0)} + E_n^{(1)}) |\phi_n^{(0)} + \phi_n^{(1)}\rangle + \dots \quad (2.169)$$

Inserting this in (2.167) gives

$$\begin{aligned}
& 0 + (\text{second order perturbations}) \\
&= \langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} \left[(1 - (K^{(0)} + K^{(1)}))(E_n^{(0)} + E_n^{(1)}) - (1 - K^{(0)})E_m^{(0)} - \right. \\
&\quad \left. - (\mathbf{H}_{v_0}^{(b)})^{(1)} + K^{(1)}(E_n^{(0)} + E_n^{(1)}) \right] | \phi_n^{(0)} + \phi_n^{(1)} \rangle \\
&= \langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} \left[(1 - K^{(0)})(E_n^{(0)} + E_n^{(1)}) - (1 - K^{(0)})E_m^{(0)} - (\mathbf{H}_{v_0}^{(b)})^{(1)} \right] | \phi_n^{(0)} + \phi_n^{(1)} \rangle .
\end{aligned} \tag{2.170}$$

When we now set $m = n$ this gives the sought-after expression for the first order energy shift:

$$E_n^{(1)} = \frac{\langle \phi_n^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (\mathbf{H}_{v_0}^{(b)})^{(1)} | \phi_n^{(0)} \rangle}{\langle \phi_n^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) | \phi_n^{(0)} \rangle} . \tag{2.171}$$

Setting $m \neq n$ instead produces an equation for the first order shift of the states

$$\langle \phi_m^{(0)} | [(\hat{\eta}_{v_0}^{(b)})^{(0)} (\mathbf{H}_{v_0}^{(b)})^{(1)} + E_n^{(1)} K^{(0)}] | \phi_n^{(0)} \rangle = (E_n^{(0)} - E_m^{(0)}) \langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) | \phi_n^{(1)} \rangle , \tag{2.172}$$

which yields (when in the degenerate case the left hand side is appropriately diagonalized as usual)

$$| \phi_n^{(1)} \rangle = \sum_{m \neq n} \frac{1}{E_n^{(0)} - E_m^{(0)}} \frac{\langle \phi_m^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (\mathbf{H}_{v_0}^{(b)})^{(1)} + E_n^{(1)} K^{(0)} | \phi_n^{(0)} \rangle}{\langle \phi_n^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (1 - K^{(0)}) | \phi_n^{(0)} \rangle} | \phi_n^{(0)} \rangle . \tag{2.173}$$

Both expressions are essentially those familiar from perturbation theory of elementary quantum mechanics. The appearance of the $K^{(0)}$ term is just a correction factor due to the fact that in the presence of a non-vanishing b -field the Hamiltonian must be modified (*cf.* (2.135)) by an additional term in order to commute with the time coordinate. Heuristically this is due to the fact that the Kalb-Ramond torsion modifies the parallel transport along v_0 .

We can use the special nature of our covariant Hamiltonian to write in the numerator of (2.171)

$$\begin{aligned}
(\hat{\eta}^{(b)})^{(0)} (\mathbf{H}_{v_0}^{(b)})^{(1)} | \phi^{(0)} \rangle &= (\hat{\eta}^{(b)})^{(0)} \left(\frac{i}{2} (v_0 \cdot \hat{\gamma}_+^{(b)} \mathbf{D}_-^{(b)} - v_0 \cdot \hat{\gamma}_-^{(b)} \mathbf{D}_+^{(b)}) - i \mathcal{L}_{v_0} \right)^{(1)} | \phi^{(0)} \rangle \\
&= (\hat{\eta}^{(b)})^{(0)} \left(\frac{i}{2} (v_0 \cdot \hat{\gamma}_+^{(b)} (\mathbf{D}_-^{(b)})^{(1)} - v_0 \cdot \hat{\gamma}_-^{(b)} (\mathbf{D}_+^{(b)})^{(1)}) - i (\mathcal{L}_{v_0})^{(1)} \right) | \phi^{(0)} \rangle \\
&= - \left(\frac{i}{2} (v_0 \cdot \hat{\gamma}_-^{(b)} (\mathbf{D}_-^{(b)})^{(1)} + v_0 \cdot \hat{\gamma}_+^{(b)} (\mathbf{D}_+^{(b)})^{(1)}) + (\hat{\eta}^{(b)})^{(0)} i (\mathcal{L}_{v_0})^{(1)} \right) | \phi^{(0)} \rangle ,
\end{aligned} \tag{2.174}$$

where $(\mathbf{D}_\pm^{(b)})^{(0)} | \phi^{(0)} \rangle = 0$ has been used. This expression drastically simplifies in the light cone limit:

Light cone limit. When there are two independent light-like Killing vectors p and k with $p \cdot p = 0 = k \cdot k$ and $p \cdot k = 1/2$, then v_0 is determined by one boost parameter γ :

$$v_0 := e^\gamma p - e^{-\gamma} k. \quad (2.175)$$

If $v_0^\mu g_{\mu\nu}^{(b)} v_0^\nu$ is independent of γ then in the limit $\gamma \rightarrow \infty$ the norm of any state $|\phi\rangle$ for which the expectation value $\langle \phi | p \cdot \hat{\gamma}_+^{(b)} p \cdot \hat{\gamma}_-^{(b)} | \phi \rangle \neq 0$ is dominated by this expectation value and scales as $e^{2\gamma}$. Hence expectation values $\langle \phi | A | \phi \rangle / \langle \phi | \hat{\eta}^{(b)} | \phi \rangle$ of any other operator A are in the light cone limit given by their component which scales as $e^{2\gamma}$, i.e. by $e^{2\gamma} \lim_{\gamma \rightarrow \infty} (e^{-2\gamma} A)$.

Comparison with (2.174) then shows that in the light cone limit we have

$$\langle \phi^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} (\mathbf{H}_{e^{-\gamma} v_0}^{(b)})^{(1)} | \phi^{(0)} \rangle \xrightarrow{\gamma \rightarrow \infty} - \langle \phi^{(0)} | (\hat{\eta}_{v_0}^{(b)})^{(0)} i(\mathcal{L}_{e^{-\gamma} v_0})^{(1)} | \phi^{(0)} \rangle. \quad (2.176)$$

This simplification is possible due to the special nature of the Hamiltonian, which, as discussed in §2.2.1 (p.21), differs from \mathcal{L}_{v_0} essentially only by being expressed in terms of commutators of the supercharges instead of anticommutators.

A similar simplification of the denominator of (2.171) does not occur *in general* in the light cone limit. But for instance for the application that will be discussed in §3.3 (p.45) K simply vanishes in this limit.

We have thus obtained a rather simple explicit general formula for the first order energy shift (as measured along some specified Killing vector field) of the supersymmetric system under consideration. In order to evaluate it one just needs to plug the expressions for the perturbed fields and operators discussed in §2.2.5 (p.35) into equation (2.171) (or its light cone limit (2.176)).

Although this calculation may of course become tedious, it is straightforward. In particular there is no need to deal with issues of gauge fixing and second-class constraints, which may become quite involved in non-trivial backgrounds (*cf.* §4 of [18]).

One practical problem of the method presented here, though, inevitably arises precisely due to its covariance: The shift in the covariant momentum is not (at least not generally) restricted to be parallel to the particular Killing vector chosen to represent the flow of parameter time, which is the only component measured by (2.171). This is no problem of principle, because the remaining spacelike momenta shifts can be computed in perturbation theory just as well:

The shifted momenta along Killing vectors v_i $i > 0$ other than the timelike vector v_0 are obtained by diagonalizing the first order perturbation of the matrix

$$P_{nm}^i := \frac{\langle \phi_n | (\hat{\eta}_{v_0}^{(b)}) i \mathcal{L}_{v_i} | \phi_m \rangle}{\langle \phi_n | (\hat{\eta}_{v_0}^{(b)}) | \phi_m \rangle}, \quad (2.177)$$

which involves first order shifts of the states themselves.

However, as will be discussed in §3 (p.39) in the context of a special example, one can choose adapted vielbein fields such that some states don't receive any curvature corrections themselves. For such states then formula (2.171) yields already all the desired information.

3. Curvature corrections to superstring spectra

The previous sections made use only of very general properties of supersymmetric quantum systems. In the following we specialize the formalism to the superstring and demonstrate the covariant perturbation theory §2.2.5 (p.35) by calculating curvature corrections to the superstring spectrum in the toy example of $\text{AdS}_3 \times \text{S}^3$ in its Penrose limit.

The technology to treat superstrings in the differential geometric framework presented here is discussed in detail in [7], which is the basis for the following discussion.

The rationale behind the following example calculation is analogous to that of [16], where the authors test a perturbation method for the bosonic string in $\text{AdS}_3 \times \text{S}^3$ in order to later apply it in [17] to the non-trivial $\text{AdS}_5 \times \text{S}^5$ case.

Other perturbation techniques for the superstring in light-cone gauge are presented in [18, 25]. The point of the formalism presented here is that it does not require to fix light-cone gauge nor even the presence of a lightlike Killing vector in target space, even though the latter does simplify the calculations. On the other hand, it is not yet clear how to incorporate RR-backgrounds in the present formalism (*cf.* [7]) which would be necessary for applications in AdS_5 .

We begin in §3.1 by demonstrating how the formalism developed here applies to the superstring in gravitational and Kalb-Ramond backgrounds. §3.2 (p.42) reviews some general facts related to the $\text{AdS}_3 \times \text{S}^3$ and its pp-wave Penrose limit which are then used in §3.3 (p.45) for the covariant perturbative calculation of the superstring spectrum in this background, following the methods developed in §2 (p.8). For comparison §3.4 (p.47) derives the exact spectrum and in §3.5 (p.48) the result is discussed.

3.1 Superstrings in B -field backgrounds with loop space formalism

As is shown in detail in [7], the closed superstring fits into the general framework of §2 (p.8) when the configuration space is identified with *loop space*, the space of maps from the circle S^1 into target space. This loop space is coordinatized by the embedding fields $X^\mu(\sigma) =: X^{(\mu, \sigma)}$ and the metric $G_{(\mu, \sigma)(\nu, \sigma')}(X)$ on loop space which is induced by the target space metric $g_{\mu\nu}$ is taken to be

$$G_{(\mu, \sigma)(\nu, \sigma')}(X) := g(X(\sigma)) \delta(\sigma, \sigma') . \quad (3.1)$$

On the exterior bundle over loop space there act the form creation operators $\mathcal{E}^{\dagger(\mu, \sigma)}$ and form annihilation operators $\mathcal{E}_{(\mu, \sigma)}$ (which, for finite dimensional manifolds, were denoted $\hat{c}^{\dagger\mu}$ and \hat{c}_ν , respectively, in §A.1 (p.53)), that, together with the coordinates $X^{(\mu, \sigma)}$ and their partial derivatives $\partial_{(\mu, \sigma)}^c$, have the canonical supercommutators

$$\begin{aligned} \left[\partial_{(\mu, \sigma)}^c, X^{(\nu, \sigma')} \right] &= \delta_\mu^\nu \delta(\sigma, \sigma') \\ \left\{ \mathcal{E}_{(\mu, \sigma)}, \mathcal{E}^{\dagger(\nu, \sigma')} \right\} &= \delta_\mu^\nu \delta(\sigma, \sigma') , \end{aligned} \quad (3.2)$$

with all other brackets vanishing.

Independent of the metric $g_{\mu\nu}$ on target space this metric on loop space has a *reparameterization isometry* generated by the vector field

$$K^{(\mu,\sigma)} := TX'^\mu(\sigma) , \quad (3.3)$$

where the constant T is identified with the string tension. It is this Killing vector field which, when used in (2.10), gives the fermionic generators of the super Virasoro algebra in the form of (modes of) the K -deformed exterior (co-)derivative on loop space:

$$\begin{aligned} \mathbf{d}_{K,\xi} &= \int d\sigma \xi(\sigma) \left(\mathcal{E}^{\dagger\mu}(\sigma) \partial_\mu^c(\sigma) + i\mathcal{E}_\mu(\sigma) X'^\mu(\sigma) \right) \\ \mathbf{d}_{K,\xi}^\dagger &= - \int d\sigma \xi(\sigma) \left(\mathcal{E}^\mu(\sigma) \nabla_\mu(\sigma) + i\mathcal{E}_\mu^\dagger(\sigma) X'^\mu(\sigma) \right) . \end{aligned} \quad (3.4)$$

Here ξ is any complex function on S^1 .

A Kalb-Ramond B -field background on target space induces a deformation of the loop space exterior derivatives as discussed in general terms in §2.1.3 (p.15).

So consider a 2-form B field on target space

$$B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu \quad (3.5)$$

which induces on loop space the operator

$$\begin{aligned} \mathbf{W}^{(B)}(X) &:= \frac{1}{2} B_{(\mu,\sigma)(\nu,\sigma')}(X) \mathcal{E}^{\dagger(\mu,\sigma)} \mathcal{E}^{\dagger(\nu,\sigma')} \\ &:= \int d\sigma \frac{1}{2} B_{\mu\nu}(X(\sigma)) \mathcal{E}^{\dagger\mu}(\sigma) \mathcal{E}^{\dagger\nu}(\sigma) \end{aligned} \quad (3.6)$$

with loop-space components

$$B_{(\mu,\sigma)(\nu,\sigma')}(X) = B_{\mu\nu}(X(\sigma)) \delta_{\sigma,\sigma'} . \quad (3.7)$$

Being the integral over a weight 1 object this operator is (classically) reparameterization invariant

$$[\mathcal{L}_\xi, \mathbf{W}^{(B)}] = 0 . \quad (3.8)$$

The deformations (2.47) now read (setting $T = 1$ for convenience)

$$\begin{aligned} \mathbf{d}_{K,\xi}^{(B)} &:= \exp(-\mathbf{W}^{(B)}) \mathbf{d}_{K,\xi} \exp(\mathbf{W}^{(B)}) \\ &= \mathbf{d}_{K,\xi} + [\mathbf{d}_{K,\xi}, \mathbf{W}^{(B)}] \\ &= \int d\sigma \xi(\sigma) \left(\mathcal{E}^{\dagger\mu}(\sigma) \hat{\nabla}_\mu(\sigma) + i\mathcal{E}_\mu(\sigma) X'^\mu(\sigma) \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6} H_{\alpha\beta\gamma}(X(\sigma)) \mathcal{E}^{\dagger\alpha}(\sigma) \mathcal{E}^{\dagger\beta}(\sigma) \mathcal{E}^{\dagger\gamma}(\sigma) - i \mathcal{E}^{\dagger\mu} B_{\mu\nu}(X(\sigma)) X'^{\nu}(\sigma) \Big) \\
\mathbf{d}_{K,\xi}^{\dagger(B)} & := (\mathbf{d}_K^{(B)})^{\dagger} \\
& = \exp(\mathbf{W}^{\dagger(B)}) \mathbf{d}_K^{\dagger} \exp(-\mathbf{W}^{\dagger(B)}) \\
& = - \int d\sigma \xi(\sigma) \left(\mathcal{E}^{\mu}(\sigma) \hat{\nabla}_{\mu}(\sigma) + i \mathcal{E}_{\mu}^{\dagger}(\sigma) X'^{\mu}(\sigma) \right. \\
& \quad \left. + \frac{1}{6} H_{\alpha\beta\gamma}(X(\sigma)) \mathcal{E}^{\alpha}(\sigma) \mathcal{E}^{\beta}(\sigma) \mathcal{E}^{\gamma}(\sigma) - i \mathcal{E}^{\mu} B_{\mu\nu}(X(\sigma)) X'^{\nu}(\sigma) \right).
\end{aligned} \tag{3.9}$$

(It can be checked [6] that this is indeed the same result found by canonical analysis of the action of the respective 1+1 dimensional nonlinear σ -model.)

In view of equation (2.69) a crucial property that needs to be checked is the algebra of the bosonic currents. A straightforward but tedious calculation gives the result

$$\begin{aligned}
[J_{a^{\pm}}^{\text{bos}\pm}(\sigma), J_{b^{\pm}}^{\text{bos}\pm}(\sigma')] & = -\frac{i}{T} \Big(\delta(\sigma, \sigma') f_{a^{\pm}}{}^{c^{\pm}}{}_{b^{\pm}} J_{c^{\pm}}^{\text{bos}\pm}(\sigma) \mp \delta'(\sigma, \sigma') 2G_{ab}(\sigma) \\
& \quad \mp 2\delta(\sigma, \sigma') X'^{\mu}(\sigma) \omega^{\pm}[e^{\pm}]_{\mu a^{\pm} a'^{\pm}} + \mathbf{R}_{a^{\pm} b^{\pm}}^{(h)} \Big),
\end{aligned} \tag{3.10}$$

(where $\mathbf{R}_{a^{\pm} b^{\pm}}^{(h)}$ is the torsion deformed curvature operator (2.52)). This equation holds true generally for arbitrary backgrounds with the objects $f_a{}^c{}_b(\sigma)$ being the “structure functions” of the vielbein:

$$f_a{}^c{}_b e_c := [e_a, e_b]. \tag{3.11}$$

For the special case of SWZW background fields these of course become the structure constants of the group and $\omega^{\pm}[e^{\pm}]$ vanishes (2.55), so that the $J_{a^{\pm}}^{\text{bos}\pm}$ really do satisfy the *current algebra*

$$[J_{a^{\pm}}^{\text{bos}\pm}(\sigma), J_{b^{\pm}}^{\text{bos}\pm}(\sigma')] = -i \frac{1}{T} \Big(\mp \delta(\sigma, \sigma') f_{a^{\pm}}{}^{c^{\pm}}{}_{b^{\pm}} J_{c^{\pm}}^{\text{bos}\pm}(\sigma) - \delta'(\sigma, \sigma') 2G_{a^{\pm} b^{\pm}}(\sigma) \Big). \tag{3.12}$$

This is the functional, canonical version of what is usually written as a CFT OPE (e.g. [20, 26])

$$J_a^{\text{bos}}(z) J_b^{\text{bos}}(w) = \frac{\frac{k^{\text{bos}}}{2} \eta_{ab}}{(z-w)^2} + \frac{i f_a{}^c{}_b J_c^{\text{bos}}(w)}{z-w}, \tag{3.13}$$

where here k^{bos} is the *level* of the current algebra generated by the bosonic currents and η_{ab} the Killing metric of the respective Lie group.

In order to make the relation between the functional and the CFT notation more manifest consider the modes

$$\begin{aligned} j_{a,n}^{\text{bos}} &:= -T \int d\sigma J_a^{\text{bos}-}(\sigma) e^{-in\sigma} \\ \tilde{j}_{a,n}^{\text{bos}} &:= -T \int d\sigma J_a^{\text{bos}+}(\sigma) e^{+in\sigma} \end{aligned} \quad (3.14)$$

which satisfy the algebra

$$\begin{aligned} [j_{a,m}^{\text{bos}}, j_{b,n}^{\text{bos}}] &= m 4\pi T g_{ab} \delta_{m,-n} + i f_a^c{}_b j_{c,m+n}^{\text{bos}} = m \frac{2}{\alpha'} g_{ab} \delta_{m,-n} + i f_a^c{}_b j_{c,m+n}^{\text{bos}} \\ [\tilde{j}_{a,m}^{\text{bos}}, \tilde{j}_{b,n}^{\text{bos}}] &= m 4\pi T g_{ab} \delta_{m,-n} + i f_a^c{}_b \tilde{j}_{c,m+n}^{\text{bos}} = m \frac{2}{\alpha'} g_{ab} \delta_{m,-n} + i f_a^c{}_b \tilde{j}_{c,m+n}^{\text{bos}}. \end{aligned} \quad (3.15)$$

Comparison with the algebra of the modes

$$j_{a,n}^{\text{bos}} := \oint \frac{dz}{2\pi i} j_a^{\text{bos}}(z) z^n \quad (3.16)$$

which reads

$$[j_{a,m}^{\text{bos}}, j_{b,n}^{\text{bos}}] = m \frac{k^{\text{bos}}}{2} \eta_{ab} \delta_{n,-m} + i f_a^c{}_b j_{c,m+n}^{\text{bos}} \quad (3.17)$$

yields the relation

$$g_{ab} = \frac{k^{\text{bos}} \alpha'}{4} \eta_{ab} \quad (3.18)$$

between level k^{bos} of the algebra of *bosonic* currents and the scale of the group manifold. Since the level k of the total currents is $k = k^{\text{bos}} - 2g^{\text{v}}$ this gives finally

$$g_{ab} = \frac{(k - 2g^{\text{v}}) \alpha'}{4} \eta_{ab}. \quad (3.19)$$

In summary, the above yields all the tools and information needed to apply the methods of §2 (p.8) to superstrings backgrounds that are supported by B -field flux. An example of an application in this context is the content of the next sections.

3.2 Review of $\text{AdS}_3 \times \text{S}^3$ and its Penrose limit

The supergravity solution of Q_5 D5-branes wrapped on a four-torus of volume v together with Q_1 fundamental strings parallel to the D5-branes reads ([27], §4)

$$\begin{aligned} e^{-2\Phi} &= \frac{1}{g^2} f_5^{-1} f_1 \\ H &= 2 \left(Q_5 \epsilon_3 + \frac{g^2 Q_1}{v} f_5 f_1^{-1} *_6 \epsilon_3 \right) \\ ds^2 &= f^{-1} (-dx_0^2 + dx_1^2) + f_5 (dr^2 + r^2 dS_3^2) + dT_4^2, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} f_1 &= 1 + \frac{g^2 \alpha' Q_1}{v} \frac{1}{r^2} \\ f_5 &= 1 + \alpha' Q_5 \frac{1}{r^2}. \end{aligned} \quad (3.21)$$

In the near-horizon limit, $\frac{1}{r^2} \gg 1$, the metric becomes that of $\text{AdS}_3 \times \text{S}^3 \times T^4$:

$$\begin{aligned} g^{-2} &= e^{-2\Phi} = \frac{1}{v} \frac{Q_1}{Q_5} \\ H &= 2Q_5 (\epsilon_3 + *_6 \epsilon_3) \\ ds^2 &= R^2 \left(-dx_0^2 + dx_1^2 \right) + \frac{1}{r^2} dr^2 + dS_3^2 \Big) + dT^4 \\ &= R^2 \left(\underbrace{-\cosh^2(\rho) dt^2 + d\rho^2 + \sinh^2(\rho) d\phi^2}_{=ds_{\text{AdS}_3}^2} + \underbrace{\cos^2(\theta) d\psi^2 + d\theta^2 + \sin^2(\theta) d\chi^2}_{=ds_{\text{S}^3}^2} \right) + dT^4, \end{aligned} \quad (3.22)$$

with

$$R^2 := Q_5 \alpha'. \quad (3.23)$$

The metric is that of the group manifold $\text{SL}(2, \mathbb{R}) \times \text{SU}(2) \times \text{U}(1)^4$ and the B -field provides the parallelizing torsion, so that superstrings on this background are described by an SWZW model (*cf.* §2.1.4 (p.17)).

Higher order corrections to the supergravity solutions will force the radius of AdS_3 to be slightly different from that of S^3 , as discussed below. Therefore write the metric (we will ignore the trivial T^4 -factor in the following) as

$$ds^2 = R_{\text{SL}}^2 \left(-\cosh^2(\rho) dt^2 + d\rho^2 + \sinh(\rho) d\phi^2 \right) + R_{\text{SU}}^2 \left(\cos^2(\theta) d\psi^2 + d\theta^2 + \sin(\theta) d\chi^2 \right). \quad (3.24)$$

A possible choice of (left/right)-invariant vielbein fields (following [28], eq. (9)) is

$$\begin{aligned} K_3 &:= -\frac{i}{2} \partial_t + \frac{i}{2} \partial_\phi \\ K_+ &:= \frac{1}{2} \left(e^{+i(\phi+t)} \tanh(\rho) \partial_t - i e^{+i(\phi+t)} \partial_\rho + e^{+i(\phi+t)} \coth(\rho) \partial_\phi \right) \\ K_- &:= \frac{1}{2} \left(-e^{-i(\phi+t)} \tanh(\rho) \partial_t - i e^{-i(\phi+t)} \partial_\rho - e^{-i(\phi+t)} \coth(\rho) \partial_\phi \right) \\ J_3 &:= -\frac{i}{2} \partial_\psi - \frac{i}{2} \partial_\chi \\ J_+ &:= \frac{1}{2} \left(-e^{+i(\chi+\psi)} \tan(\rho) \partial_\psi - i e^{+i(\chi+\psi)} \partial_\theta + e^{+i(\chi+\psi)} \cot(\rho) \partial_\chi \right) \\ J_- &:= \frac{1}{2} \left(e^{-i(\chi+\psi)} \tan(\rho) \partial_\psi - i e^{-i(\chi+\psi)} \partial_\theta - e^{-i(\chi+\psi)} \cot(\rho) \partial_\chi \right). \end{aligned} \quad (3.25)$$

$$\begin{aligned}
K_3 &:= -\frac{i}{2}\partial_t + \frac{i}{2}\partial_\phi \\
K_+ &:= \frac{1}{2}\left(e^{-i(\phi-t)}\tanh(\rho)\partial_t - ie^{-i(\phi-t)}\partial_\rho - e^{-i(\phi-t)}\coth(\rho)\partial_\phi\right) \\
K_- &:= \frac{1}{2}\left(-e^{+i(\phi-t)}\tanh(\rho)\partial_t - ie^{+i(\phi-t)}\partial_\rho + e^{+i(\phi-t)}\coth(\rho)\partial_\phi\right) \\
J_3 &:= -\frac{i}{2}\partial_\psi + \frac{i}{2}\partial_\chi \\
J_+ &:= \frac{1}{2}\left(-e^{-i(\chi-\psi)}\tan(\rho)\partial_\psi - ie^{-i(\chi-\psi)}\partial_\theta - e^{-i(\chi-\psi)}\cot(\rho)\partial_\chi\right) \\
J_- &:= \frac{1}{2}\left(e^{+i(\chi-\psi)}\tan(\rho)\partial_\psi - ie^{+i(\chi-\psi)}\partial_\theta + e^{+i(\chi-\psi)}\cot(\rho)\partial_\chi\right). \quad (3.26)
\end{aligned}$$

These vectors are normalized so as to have the standard non-vanishing Lie brackets:

$$\begin{aligned}
[K_3, K_\pm] &= \pm K_\pm \\
[K_+, K_-] &= -2K_3 \\
[J_3, J_\pm] &= \pm J_\pm \\
[J_+, J_-] &= +2J_3 \quad (3.27)
\end{aligned}$$

This fixes their inner products with respect to ds^2 to

$$\begin{aligned}
K_3 \cdot K_3 &= \frac{R_{\text{SL}}^2}{4} \\
K_+ \cdot K_- &= -2\frac{R_{\text{SL}}^2}{4} \\
J_3 \cdot J_3 &= -\frac{R_{\text{SU}}^2}{4} \\
J_+ \cdot J_- &= -2\frac{R_{\text{SU}}^2}{4}. \quad (3.28)
\end{aligned}$$

This is $\pm\frac{1}{4}R_{\text{SL/SU}}^2$ times the Killing metric on the two group manifolds.⁷ According to (3.19) this means that the level of the associated algebra of *total* currents is

$$k = R_{\text{SL}}^2/\alpha' - 2$$

⁷With $e_0 := K_3$, $e_1 := K_+$, \dots and the structure constants $f_a{}^c{}_b$ defined by $[e_a, e_b] = f_a{}^c{}_b e_c$ we have

$$-\frac{1}{2}f_a{}^r{}_s f_b{}^s{}_r = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{bmatrix}. \quad (3.29)$$

$$= R_{\text{SU}}^2/\alpha' + 2, \quad (3.30)$$

which is, up to a small correction, proportional to the size of spacetime in units of the string scale. For the calculation of the exact string spectrum on $\text{AdS}_3 \times \text{S}^3$ (this is discussed in §3.4 (p.47) below) one needs the quadratic Casimir

$$\begin{aligned} C &:= -\eta^{ab}e_a e_b \\ &= -K_3(K_3 + 1) + K_-K_+ + J_3(J_3 + 1) + J_-J_+. \end{aligned} \quad (3.31)$$

A particularly interesting further limit is the *Penrose limit* of the $\text{AdS}_3 \times \text{S}^3$ background (see e.g. [29]). It is obtained by concentrating on the vicinity of a lightlike geodesic going around the equator of the S^3 factor, i.e. one with momentum proportional to $K_3 \pm J_3$.

In order to find the background structure in this limit introduce the following vielbein basis adapted to this geodesic motion:

$$\begin{aligned} F &:= \frac{1}{k}(J_3 - K_3) \\ J &:= J_3 + K_3 \\ P_1 &:= \frac{1}{\sqrt{k}}K_+ \\ P_1^* &:= \frac{1}{\sqrt{k}}K_- \\ P_2 &:= \frac{1}{\sqrt{k}}J_+ \\ P_2^* &:= \frac{1}{\sqrt{k}}J_- . \end{aligned} \quad (3.32)$$

Since their non-vanishing commutators are

$$\begin{aligned} [J, P_i] &= P_i \\ [J, P_i^*] &= -P_i^* \\ [P_1, P_1^*] &= F - \frac{1}{k}J \\ [P_2, P_2^*] &= F + \frac{1}{k}J \\ [F, P_i^{(*)}] &= \pm \frac{1}{k}P_i^{(*)} \end{aligned} \quad (3.33)$$

one sees that in the Penrose limit $J_3 - K_3 \approx k \rightarrow \infty$ with $J_3 \approx -K_3$ the Lie algebra *contracts* to that of the so-called extended Heisenberg group H_6 which describes a pp-wave background [29].

3.3 Covariant perturbative calculation of the superstring spectrum

Our aim is to use the perturbation theory of §2 (p.8) to calculate (in the same spirit as [16] but using covariant techniques and superstrings) the correction to the string

spectrum in the small parameter $1/k$. That is, we start with the exact spectrum of superstrings on the H_6 pp-wave background and then turn on curvature corrections turning the pp-wave background into the true $\text{AdS}_3 \times \text{S}^3$ geometry.

The calculation involves computing the various perturbed quantities discussed in §2.2.4 (p.31). Most importantly, one finds for the metric in the adapted vielbein basis (3.32) the expansion

$$\eta_{ab} = \frac{\alpha'}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} + \frac{\alpha'}{k} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} + \mathcal{O}(1/k^2) . \quad (3.34)$$

An examination of the B -deformed vielbein field (2.126) shows that the calculation simplifies when $v_0 := e^\gamma F - e^{-\gamma} J$ for $\gamma \rightarrow \infty$ is chosen as the timelike Killing vector (2.70), since then the correction operator K (2.138) *vanishes*, $K = 0$ and we can make use of formula (2.176) to evaluate the first order shift of string energy as measured along the Killing vector v_0 by computing the expectation value of the first order shift in the loop-space Lie derivative along F . By using equation (2.63) one finds that this Lie derivative is just the sum of the left- and right-moving total SWZW *currents* along F (2.61):

$$i(\mathcal{L}_F) = -i(J_{F,0}^+ + J_{F,0}^-) . \quad (3.35)$$

The perturbation in this loop-space Lie derivative is most conveniently computed using formula (A.86) in the appendix. One finds

$$i(\mathcal{L}_F)^{(1)} = -\frac{1}{k} \frac{1}{\alpha'} \int d\sigma \left(\Gamma_{P_1,+} \Gamma_{P_1^*,+} - \Gamma_{P_2,+} \Gamma_{P_2^*,+} - \Gamma_{P_1,-} \Gamma_{P_1^*,-} + \Gamma_{P_2,-} \Gamma_{P_2^*,-} \right) . \quad (3.36)$$

According to formula (2.176) the energy shifts that we are looking for are the expectation values of (3.36) in the unperturbed states. Obviously (3.36) is just a kind of fermion number-operator. To be more precise, let $N_{\text{SL}}^{\text{fer}}$ be the number of Γ_{P_1} excitations minus the number of $\Gamma_{P_1^*}$ excitations of the string, and similarly let $N_{\text{SU}}^{\text{fer}}$ be the number of Γ_{P_2} excitations minus the number of $\Gamma_{P_2^*}$ excitations for both the left- and the right-moving sector. This is a measure for the fermionic contribution to the angular momentum of the string state with respect to K_3 and J_3 (*cf.* [29] and see also the discussion §3.4 (p.47) below). By explicitly constructing the bosonic and fermionic physical DDF states for Type II strings in the pp-wave limit of $\text{AdS}_3 \times \text{S}^3$ (a calculation that closely follows [29] and will therefore not be given here) one checks that our unperturbed states are indeed eigenstates with respect to $\hat{N}_{\text{SL}}^{\text{fer}}$ and $\hat{N}_{\text{SU}}^{\text{fer}}$.

This means that we can finally write down the expectation values of (3.36) in the unperturbed states, which are nothing but the energy shifts $E^{(1)}$ that we are looking for, as

$$E^{(1)} = \frac{1}{k} \left(N_{\text{SL}}^{\text{fer}} - N_{\text{SU}}^{\text{fer}} \right) . \quad (3.37)$$

This is the result of our covariant perturbative calculation of the spectrum of Type II superstrings around the pp-wave limit of $\text{AdS}_3 \times \text{S}^3$.

In order to check this result, the next section discusses the calculation of the *exact* superstring spectrum on $\text{AdS}_3 \times \text{S}^3$. The result is further discussed in §3.5 (p.48)

3.4 Exact calculation of the spectrum

In the following a generalization of the discussion in §4 of [16] is given, calculating the lightcone energy of superstrings on $\text{AdS}_3 \times \text{S}^3$ to all orders in $1/k \approx \alpha'/R^2$.

We first consider the bosonic string (and concentrate on one chirality sector for notational brevity): Let $-h(h+1) + j(j+1)$ be the eigenvalue of the Casimir $\eta_{ab} J_0^a J_0^b$ of the $\text{SL}(2, \mathbb{R}) \times \text{SU}(2)$ current algebra and let $N \in \mathbb{N}$ be the level of a given state. Then the L_0 Virasoro constraint on this state reads

$$-\frac{h(h+1)}{k} + \frac{j(j+1)}{k} + N = a \quad (3.38)$$

for a given normal ordering constant a .

The eigenvalues of h^3 and j^3 of the zero modes of K_0^3 and J_0^3 can be written as

$$\begin{aligned} h^3 &= h + N'_{\text{SL}} \\ j^3 &= j + N'_{\text{SU}} , \end{aligned} \quad (3.39)$$

where, for instance, N'_{SU} grows by one for every J_{-n}^+ (bosonic current) excitation and is reduced by one for every J_{-n}^- excitation (due to $[J_0^3, J_n^\pm] = \pm J_n^\pm$ and $[J_0^3, \psi_n^\pm] = \pm \psi_n^\pm$).

The characteristic lightlike momenta of the H_6 model are, according to (3.32), the light cone energy H associated with the vector field J and transversal momentum p_- associated with the vector field F :

$$\begin{aligned} H &= h^3 + j^3 \\ p_- &= \frac{1}{k} (h^3 - j^3) . \end{aligned} \quad (3.40)$$

Using the physical state condition (3.38) we want to express these momenta as functions of each other and of the transverse excitations:

$$\begin{aligned} H &= H(p_-, N, N') \\ p_- &= p_-(H, N, N') . \end{aligned} \quad (3.41)$$

Solving (3.38) for h and picking the positive solution yields

$$h = \frac{-1 + \sqrt{1 + 4j + 4j^2 - 4ak + 4kn}}{2}. \quad (3.42)$$

Furthermore, equations (3.40) solved for j give, respectively:

$$\begin{aligned} j &= \frac{H + H^2 - k(N - a) + (N'_{\text{SL}} + N'_{\text{SU}})(1 + 2H + N'_{\text{SL}} + N'_{\text{SU}})}{2(1 + H + N'_{\text{SL}} + N'_{\text{SU}})} \\ j &= \frac{k(a - N - p_-) + N'_{\text{SU}} - N'_{\text{SL}} + (kp_- + N'_{\text{SU}} - N'_{\text{SL}})^2}{2(N'_{\text{SL}} - N'_{\text{SU}} - kp_-)} \end{aligned} \quad (3.43)$$

Inserting k and j in (3.40) yields the simple result

$$H = -1 + N'_{\text{SL}} + N'_{\text{SU}} + \frac{N - a}{p_- - (N'_{\text{SL}} - N'_{\text{SU}})/k} \quad (3.44)$$

$$\Leftrightarrow p_- = \frac{N - a}{1 + H - N'_{\text{SL}} - N'_{\text{SU}}} + \frac{N'_{\text{SL}} - N'_{\text{SU}}}{k}. \quad (3.45)$$

When the expression for H is expanded to first order in $1/k$ and $N'_{\text{SL}} = 0$ we get the result known from [16]. Note that the series for p_- stops already after the first order.

The generalization to the superstring is immediate: The j and h quantum numbers are now those associated with the *bosonic* currents J_{bos}^a but for the light cone generators we have to use the total currents J_{tot}^a , since these act as Lie derivatives, cf. (2.63). The total currents are just the sum of the bosonic and the fermionic currents (2.61) We therefore have simply

$$\begin{aligned} h_3 &= h_3^{\text{bos}} + h_3^{\text{fer}} + N'_{\text{SL}} \\ h_3 &= h_3^{\text{bos}} + h_3^{\text{fer}} + N'_{\text{SU}} \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} N'_{\text{SL}} &= N_{\text{SL}}^{\text{bos}} + N_{\text{SL}}^{\text{fer}} \\ N'_{\text{SU}} &= N_{\text{SU}}^{\text{bos}} + N_{\text{SU}}^{\text{fer}}. \end{aligned} \quad (3.47)$$

In summary, the first order perturbation $(p_-)^{(1)}$ of p_- (the momentum associated with the lightlike Killing vector F) for fixed H is

$$(p_-)^{(1)} = \frac{1}{k} (N'_{\text{SL}} - N'_{\text{SU}}). \quad (3.48)$$

3.5 Discussion of the perturbative result

Comparison of the perturbative result (3.37) with the exact calculation (3.48) seems to show that the covariant perturbation theory reproduces the fermionic contribution exactly, while it seemingly misses the bosonic one completely. A little reflection shows

however that the comparison of these results has to take into account the following subtlety:

In the perturbative calculations which use the lightcone gauge (as in [16]) one can fix the longitudinal momentum p_- by hand while turning on the perturbation and calculate the shift in lightcone energy H for fixed p_- as in equation (3.44).

However, the covariant framework that has been presented here does not fix any gauge and in particular does not fix any of the lightcone momenta. This means that when the background is perturbed, the states are free to acquire shifts in H *or* in p_- *or* in both. But only the combination of both H and p_- has invariant meaning, which is encoded in the relations (3.44) and (3.45) that express p_- as a function of H and the excitations of the string, or vice versa (3.41).

For a complete covariant perturbative result one therefore would need to compute not only the energy shift along v_0 (i.e. p_- in the above case), but also the shift in the other longitudinal momentum (H). This has already been discussed at the end of §2.2.5 (p.35), where it was pointed out that the computation of the shift in the second longitudinal momentum is tedious, because it requires knowledge of the first order perturbation of the states themselves.

Here we shall be content with arguing that for fermionic states no shift in H occurs, according to (2.177). The reason is that on the one hand side one can calculate the shift in the loop-space Lie derivative along the vector field J (which measures the momentum H according to (3.40)) to be purely bosonic, having vanishing expectation value in the unperturbed fermionic states. Furthermore, no shift in the fermionic *states* can be expected to give any contribution to a shift in H due to (2.177), because the fermionic states are created by the $\Gamma_{P_i^{(*)}}$ oscillators together with longitudinal terms that ensure physicality (as in the DDF construction). But since the $\Gamma_{P_i^{(*)}}$ are defined with respect to the invariant vielbein (3.32) they receive no correction in $1/k$, according to equation (2.146). A fermionic state in the pp-wave limit created by a given mode of the operator $\Gamma_{P_i^{(*)}}$ should hence flow to a state of the full AdS background created by the same operator $\Gamma_{P_i^{(*)}}$ possibly accompanied by different longitudinal excitations. But these do not contribute to any inner products.

On the other hand, bosonic states are created by the bosonic currents, which, according to (2.62) are rather complicated expressions involving products of the background metric and background b -field with the elementary fields (2.146). Therefore nothing can be said in general about the first order shift for H of the bosonic states, while $H^{(1)}$ for the fermionic states should vanish.

From these considerations it follows that equation (3.37) gives the shift of p_- for fixed H for fermionic states, while it tells us nothing about the shift of H for bosonic states. In conclusion then the perturbative result (3.37) gives the correct result (3.44) for all the cases where it applies, which are the fermionic states. The other cases may be treated, too, in principle, but require much more computational effort, since they

require a computation of the first order shift in the states themselves. This is the price to be paid due to working in a fully covariant framework where no worldsheet gauge is fixed. Hence we find a partial result using a relatively elegant calculation, while the full result requires tedious work.

What then is the point of using the covariant perturbative calculation presented here, if, as in the example discussed, the calculation of the full result is more involved than the corresponding calculation using lightcone gauge? There are two answers:

First, one should note that the fermionic spectrum which we obtained easily is, according to §3.4 (p.47), an exact mirror image of the bosonic spectrum. As long as one knows that this is the case the calculation of the energy shift of the fermionic states, which is simple in our framework, already yields the full information about energy shifts of all states.

Second, the motivation for the construction of the perturbation scheme developed in §2 (p.8) was to find a method that is more generally applicable than the methods requiring lightcone gauge are, since no lightlike Killing vector is required on target space. It is almost inevitable that the more general method is more involved than the one which is adapted to special cases of high symmetry.

A more general assessment of what has been accomplished here is given in the following section.

4. Conclusion

It has been shown that covariant Hamiltonian evolution operators can be constructed in relativistic supersymmetric quantum (field) theories for a large class of interesting backgrounds, by reformulating these theories as generalized Dirac-Kähler systems on the exterior bundle over their bosonic configuration space.

The crucial insight was that any system of supersymmetry constraints $\mathbf{D}_\pm |\psi\rangle = 0$ can equivalently be rewritten as a Schrödinger equation generating evolution along a time parameter together with a constraint on hypersurfaces orthogonal to that time parameter. In various guises this construction is well familiar from both the Dirac particle as well as the classical Maxwell field. It is no coincidence that these two systems are related to the supersymmetric formalism discussed here, since they can be regarded as two sectors of the NSR *superparticle*, i.e. the point particle limit of the NSR superstring. We have shown how to incorporate both sectors in one coherent formalism and how to generalize this to backgrounds with a non-vanishing 2-form Kalb-Ramond field and hence in particular to supersymmetric Wess-Zumino-Novikov-Witten models.

In doing so we made use of the fact that the supersymmetry constraints for such backgrounds can be obtained from those for trivial backgrounds by an algebra homomorphism which generalizes the deformations considered by Witten in [1]. This

is crucial, because, as we have shown, by appropriately applying similar deformations to all operators which appear in the construction of the covariant Hamiltonian for trivial backgrounds one obtains the covariant Hamiltonian for the non-trivial background.

One subtlety that remains is that the Hamiltonian obtained this way, though satisfying a formal Schrödinger equation, in general no longer commutes with the time parameter coordinate. But this can be fixed by appropriately subtracting the offending terms consistently on both sides of the Schrödinger equation.

When all this is done it is rather straightforward to adapt the familiar techniques of quantum mechanical perturbation theory: After dealing with the indefiniteness of the Hodge inner product by employing a Krein space operator and after taking into account the above mentioned correction to the Hamiltonian operator one obtains an equation for the first order energy shift that is formally very similar to the one derived in elementary quantum mechanics.

Because it is of importance for the application presented in §3 (p.39) we finally considered the case where the Hamiltonian evolution is along a (almost) lightlike vector. It turns out that the special nature of the Hamiltonian considered here, together with the presence of that Krein space operator, leads to a considerable simplification of the formula for the first order energy shift in this case.

It should be noted, that this does not involve fixing any gauge, whatsoever, in particular this is not related to fixing a light cone gauge. The methods presented here are equally valid in backgrounds which do not possess any lightlike Killing vectors at all. This makes them interesting for the study of superstring theory in arbitrary nontrivial backgrounds.

As demonstrated in §3.1 (p.39) and [7], the machinery developed here carries over to the case where the underlying manifold is loop space, the configuration space of the string. The calculation presented in §3.3 (p.45) demonstrates how to apply the above perturbation scheme to perturbatively calculate the first order curvature correction for superstrings close to the pp-wave limit of $\text{AdS}_3 \times S^3$, as was done for the bosonic string in light cone gauge in [16].

It turns out that the calculation of fermionic states (those created by fermionic worldsheet oscillators from the ground state) by our method fully profits from the elegance of the covariant approach, while the first order spectrum of bosonic states requires knowledge of the first order shift in the states themselves, a fact that makes any direct calculation much more tedious. This has been discussed in detail in §3.5 (p.48).

The natural next step would be to apply our formalism to superstring spectra on $\text{AdS}_5 \times S^5$ (*cf.* [18]). This requires the as yet unknown incorporation of RR-background fields into the framework of §2 (p.8). As is well known, RR-backgrounds are almost impossible to handle in terms of σ -models and Lagrangian formalism. Therefore it would be interesting to further analyze the deformation mechanism

of §2.1.2 (p.11). Possibly this way backgrounds can be incorporated that defy a Lagrangian description. First steps in this direction are discussed in [7] and [8].

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A. Differential geometry in terms of operators on the exterior bundle

A.1 Creation/Annihilation and Clifford algebra

Consider a semi-Riemannian manifold (\mathcal{M}, g) of dimension D with metric g , which has signature $(D - s, s)$. On the space $\Omega(\Lambda(\mathcal{M}))$ (which we take to be complexified) of a suitable class of sections of the exterior bundle $\Lambda(\mathcal{M})$ (the bundle of differential forms of arbitrary degree) over this manifold, we have the operators $\hat{c}^{\dagger\mu}$ of exterior multiplication, defined by

$$\hat{c}^{\dagger\mu}\omega := dx^\mu \wedge \omega, \quad \Omega(\Lambda(\mathcal{M})) \ni \omega = \omega_{(0)} + \omega_{\mu_1} dx^{\mu_1} + \omega_{\mu_1\mu_2} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \quad (\text{A.1})$$

With respect to the usual Hodge inner product $\langle \cdot | \cdot \rangle$ on $\Omega(\Lambda(\mathcal{M}))$,

$$\begin{aligned} \langle \alpha | \beta \rangle &= \int_{\mathcal{M}} \bar{\alpha} \wedge \star \beta \\ &:= p! \int_{\mathcal{M}} \sqrt{g} \bar{\alpha}_{\mu_1\mu_2\dots} \beta^{\mu_1\mu_2\dots} d^D x \end{aligned} \quad (\text{A.2})$$

(where $\bar{\alpha}$ is the complex conjugate of α), which defines the Hodge- \star operator, their adjoints are $\hat{c}^\mu := (\hat{c}^{\dagger\mu})^\dagger$, and both together satisfy the canonical anticommutation relations (CAR)

$$\begin{aligned} \{\hat{c}^{\dagger\mu}, \hat{c}^{\dagger\nu}\} &= 0 \\ \{\hat{c}_\mu, \hat{c}_\nu\} &= 0 \\ \{\hat{c}_\mu, \hat{c}^{\dagger\nu}\} &= \delta_\mu^\nu. \end{aligned} \quad (\text{A.3})$$

With the linear combinations

$$\hat{\gamma}_\pm^\mu := \hat{c}^{\dagger\mu} \pm \hat{c}^\mu \quad (\text{A.4})$$

this is isomorphic to the Clifford algebra

$$\begin{aligned} \{\hat{\gamma}_\pm^\mu, \hat{\gamma}_\mp^\nu\} &= 0 \\ \{\hat{\gamma}_\pm^\mu, \hat{\gamma}_\pm^\nu\} &= \pm 2g^{\mu\nu}. \end{aligned} \quad (\text{A.5})$$

Every element of the Clifford algebra is mapped to a differential form by the *symbol map*

$$(\omega_{(0)} + \omega_{\mu_1} \hat{\gamma}_\pm^{\mu_1} + \omega_{[\mu_1, \mu_2]} \hat{\gamma}_\pm^{\mu_1} \hat{\gamma}_\pm^{\mu_2} + \dots) |1\rangle = \omega_{(0)} + \omega_{\mu_1} dx^{\mu_1} + \omega_{[\mu_1, \mu_2]} dx^{\mu_1} \wedge dx^{\mu_2} + \dots \quad (\text{A.6})$$

where $|1\rangle$ denotes the constant unit 0-form. The *local* inner product $\langle \alpha | \beta \rangle_{\text{loc}}$ is defined by

$$\langle \alpha | \beta \rangle = \int_{\mathcal{M}} \langle \alpha | \beta \rangle_{\text{loc}} \sqrt{g} d^D x, \quad (\text{A.7})$$

and also serves as the projection on Clifford 0-vectors, i.e.

$$\langle 1 | \left(\omega_{(0)} + \omega_{\mu_1} \hat{\gamma}_{\pm}^{\mu_1} + \omega_{[\mu_1, \mu_2]} \hat{\gamma}_{\pm}^{\mu_1} \hat{\gamma}_{\pm}^{\mu_2} + \dots \right) | 1 \rangle_{\text{loc}} := \omega_{(0)} . \quad (\text{A.8})$$

It has the cyclic property

$$\langle 1 | \hat{\gamma}_{\pm}^{a_1} \hat{\gamma}_{\pm}^{a_2} \dots \hat{\gamma}_{\pm}^{a_p} | 1 \rangle_{\text{loc}} = \langle 1 | \hat{\gamma}_{\pm}^{a_2} \dots \hat{\gamma}_{\pm}^{a_p} \hat{\gamma}_{\pm}^{a_1} | 1 \rangle_{\text{loc}} . \quad (\text{A.9})$$

Using a vielbein field e^a_{μ} on \mathcal{M} we write the ONB frame version of these operators as

$$\begin{aligned} \hat{e}^{\dagger a} &:= e^a_{\mu} \hat{c}^{\dagger \mu} \\ \hat{e}^a &:= e^a_{\mu} \hat{c}^{\mu} \\ \hat{\gamma}_{\pm}^a &:= e^a_{\mu} \hat{\gamma}_{\pm}^{\mu} . \end{aligned} \quad (\text{A.10})$$

The *number operator*, which measures the degree of a differential form, is defined by

$$\begin{aligned} \hat{N} &= \hat{c}^{\dagger \mu} \hat{c}_{\mu} \\ &= \hat{e}^{\dagger a} \hat{e}_a . \end{aligned} \quad (\text{A.11})$$

Note that

$$[\hat{N}, \hat{\gamma}_{\pm}^{\mu}] = \hat{\gamma}_{\mp}^{\mu} . \quad (\text{A.12})$$

A shifted version of this operator, with symmetrized spectrum, is

$$\frac{1}{2} \hat{\gamma}_{-}^a \hat{\gamma}_{+,a} = \hat{N} - D/2 . \quad (\text{A.13})$$

Often it is convenient to use a slightly modified version of the Hodge- \star operator, namely:

$$\bar{\star} := i^{D(D-1)/2+s} \begin{cases} \hat{\gamma}_{-}^{a=0} \hat{\gamma}_{-}^{a=1} \dots \hat{\gamma}_{-}^{a=D-1} & \text{if } D \text{ is even} \\ \hat{\gamma}_{+}^{a=0} \hat{\gamma}_{+}^{a=1} \dots \hat{\gamma}_{+}^{a=D-1} & \text{if } D \text{ is odd} \end{cases} , \quad (\text{A.14})$$

which is conveniently normalized so as to satisfy the relations

$$(\bar{\star})^{\dagger} = (-1)^s \bar{\star} \quad (\text{A.15})$$

$$(\bar{\star})^2 = 1 \quad (\text{A.16})$$

$$\bar{\star} \hat{e}^{\dagger a} = \hat{e}^a \bar{\star} . \quad (\text{A.17})$$

It is related to the Hodge- \star via

$$\bar{\star} = \star i^{D(D-1)/2+s} (-1)^{\hat{N}(\hat{N}+1)/2+D} . \quad (\text{A.18})$$

We note here the simple but important relation

$$\begin{aligned}\bar{\star}\hat{N} &= \hat{e}_a\hat{e}^{\dagger a}\bar{\star} \\ &= (D - \hat{N})\bar{\star}.\end{aligned}\tag{A.19}$$

For $s > 0$ the inner product $\langle \cdot | \cdot \rangle$ is indefinite. Assume $s = 1$, which is the case of interest here, and $\{\hat{e}^0, \hat{e}^{\dagger 0}\} = -1$. Then the operator

$$\begin{aligned}\hat{\eta} &:= \hat{e}^{\dagger 0}\hat{e}^0 - \hat{e}^0\hat{e}^{\dagger 0} \\ &= \hat{\gamma}_-^{a=0}\hat{\gamma}_+^{a=0},\end{aligned}\tag{A.20}$$

(which is self-adjoint with respect to $\langle \cdot | \cdot \rangle$: $\hat{\eta}^\dagger = \hat{\eta}$) swaps the spurious sign, and the modified inner product

$$\langle \cdot | \cdot \rangle_{\hat{\eta}} := \langle \cdot | \hat{\eta} \cdot \rangle\tag{A.21}$$

is positive definite and indeed a scalar product. The adjoint of an operator A with respect to $\langle \cdot | \cdot \rangle_{\hat{\eta}}$ will be written $A^{\dagger_{\hat{\eta}}}$ and is given by

$$\begin{aligned}A^{\dagger_{\hat{\eta}}} &= (\hat{\eta}A\hat{\eta}^{-1}) \\ &= \hat{\eta}^{-1}A^{\dagger}\hat{\eta}.\end{aligned}\tag{A.22}$$

(The term $\hat{\eta}^{-1}$ is here not evaluated further to allow for slightly more general $\hat{\eta}$ that will be discussed in §2.2.1 (p.21), *cf.* (2.81).)

A.2 Differential operators

Let $\hat{\nabla}_\mu$, which is the *covariant derivative operator* with respect to the Levi-Civita-connection $\Gamma_\mu^\alpha{}_\beta$ of $g_{\mu\nu}$, be defined by

$$\begin{aligned}[\hat{\nabla}_\mu, f] &= (\partial_\mu f), \quad f \in \Lambda^0(\mathcal{M}) \\ [\hat{\nabla}_\mu, \hat{c}^{\dagger\alpha}] &= -\Gamma_\mu^\alpha{}_\beta \hat{c}^{\dagger\beta}.\end{aligned}\tag{A.23}$$

If $\omega_\mu{}^a{}_b$ is the Levi-Civita connection in the orthonormal vielbein frame,

$$\omega_\mu{}^a{}_b := e^a{}_\alpha (\delta^\alpha{}_\beta \partial_\mu + \Gamma_\mu^\alpha{}_\beta) (e^{-1})^\beta{}_b,\tag{A.24}$$

then the last line is equivalent to

$$[\hat{\nabla}_\mu, \hat{e}^{\dagger a}] = -\omega_\mu{}^a{}_b \hat{e}^{\dagger b}.\tag{A.25}$$

This way one has:

$$\begin{aligned}\hat{\nabla}_\mu (\omega_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}) &= (\nabla_\mu \omega_{\alpha_1 \dots \alpha_p}) dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \\ &= (\nabla_{[\mu} \omega_{\alpha_1 \dots \alpha_p]}) dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}.\end{aligned}\tag{A.26}$$

Usually one also identifies the operator version of the *connection 1-form*

$$\omega^a{}_b := \hat{c}^{\dagger\mu} \omega_\mu{}^a{}_b. \quad (\text{A.27})$$

The commutator of the covariant derivative operators with themselves gives the *Riemann curvature operator*:

$$\begin{aligned} [\hat{\nabla}_\mu, \hat{\nabla}_\nu] &:= \mathbf{R}_{\mu\nu} \\ &:= R_{\mu\nu\alpha\beta} \hat{c}^{\dagger\alpha} \hat{c}^\beta. \end{aligned} \quad (\text{A.28})$$

From the covariant derivative operator one can construct two flavors of partial derivative operators, distinguished by which of the basis forms they respect as constants, i.e. with which set of basis forms they commute. Introducing the operators

$$\begin{aligned} \partial_\mu &:= \hat{\nabla}_\mu + \omega_\mu{}^a{}_b \hat{c}^{\dagger b} \hat{e}_a \\ \partial_\mu^c &:= \hat{\nabla}_\mu + \Gamma_\mu{}^\alpha{}_\beta \hat{c}^{\dagger\beta} \hat{e}_\alpha, \end{aligned} \quad (\text{A.29})$$

which are, according to (A.24), related as

$$\partial_\mu^c = \partial_\mu - e^a{}_\alpha (\partial_\mu e^\alpha{}_b) \hat{c}^{\dagger b} \hat{e}_a, \quad (\text{A.30})$$

one finds

$$\begin{aligned} [\partial_\mu, f] &= (\partial_\mu f), \quad f \in \Lambda^0(\mathcal{M}) \\ [\partial_\mu, \hat{e}^{\dagger a}] &= 0 \\ [\partial_\mu, \hat{e}_a] &= 0. \end{aligned} \quad (\text{A.31})$$

and

$$\begin{aligned} [\partial_\mu^c, f] &= (\partial_\mu f), \quad f \in \Lambda^0(\mathcal{M}) \\ [\partial_\mu^c, \hat{c}^{\dagger\alpha}] &= 0 \\ [\partial_\mu^c, \hat{c}_\alpha] &= 0. \end{aligned} \quad (\text{A.32})$$

(Note the position of the indices in the last two lines.) By acting with the partial derivative operators on an arbitrary form in a given basis one also verifies that for both the expected relations

$$\begin{aligned} [\partial_\mu, \partial_\nu] &= 0 \\ [\partial_\mu^c, \partial_\nu^c] &= 0 \end{aligned} \quad (\text{A.33})$$

hold. Using (A.29), (A.31), and (A.32) it is now easy to establish the transformation properties of all creators and annihilators starting from (A.23):

$$[\hat{\nabla}_\mu, \hat{e}^{\dagger a}] = +\omega_\mu{}^b{}_a \hat{e}^{\dagger b}$$

$$\begin{aligned}
[\hat{\nabla}_\mu, \hat{e}^a] &= -\omega_\mu{}^a{}_b \hat{e}^b \\
[\hat{\nabla}_\mu, \hat{e}_a] &= +\omega_\mu{}^b{}_a \hat{e}_b \\
[\hat{\nabla}_\mu, \hat{c}^\dagger_\alpha] &= +\Gamma_\mu{}^\beta{}_\alpha \hat{c}^\dagger_\beta \\
[\hat{\nabla}_\mu, \hat{c}^\alpha] &= -\Gamma_\mu{}^\beta{}_\alpha \hat{c}^\alpha \\
[\hat{\nabla}_\mu, \hat{c}_\alpha] &= +\Gamma_\mu{}^\beta{}_\alpha \hat{c}_\beta.
\end{aligned} \tag{A.34}$$

That is, all basis operators transform as they should according to the index they carry.

Note that in particular we can now write

$$\begin{aligned}
\hat{\nabla}_\mu &= \partial_\mu - \omega_\mu{}^a{}_b \hat{e}^{\dagger b} \hat{e}_a \\
&= \partial_\mu + \omega_{\mu ab} \hat{e}^{\dagger a} \hat{e}^{\dagger b} \\
&= \partial_\mu + \frac{1}{4} \omega_{\mu ab} (\hat{\gamma}_+^a \hat{\gamma}_+^b + \hat{\gamma}_-^a \hat{\gamma}_-^b).
\end{aligned} \tag{A.35}$$

Another useful fact is that ∂_μ and $\hat{\nabla}_\mu$ commute with the duality operation:

$$\begin{aligned}
[\partial_\mu, \bar{\star}] &= 0 \\
[\hat{\nabla}_\mu, \bar{\star}] &= 0,
\end{aligned} \tag{A.36}$$

which follows straightforwardly by using the respective definitions.

With respect to the Hodge inner product the adjoint of ∂_μ is

$$(\partial_\mu)^\dagger = -\frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|}. \tag{A.37}$$

On the other hand the operator ∂_μ^c satisfies no such simple formula. Using the antisymmetry of $\omega_{\mu ab} = \omega_{\mu[ab]}$ one finds from (A.37) and (A.29) the analogous relation

$$(\hat{\nabla}_\mu)^\dagger = -\frac{1}{\sqrt{|g|}} \hat{\nabla}_\mu \sqrt{|g|}. \tag{A.38}$$

Next, it is of interest to have differential operators without free indices, which map forms to forms. Such are obtained by contracting $\hat{\nabla}_\mu$ with some Grassmann or Clifford operator:

Exterior derivative. The *exterior derivative* is defined by

$$\mathbf{d} := \hat{c}^{\dagger\mu} \hat{\nabla}_\mu. \tag{A.39}$$

Due to the special symmetry of the Levi-Civita connection in the coordinate basis, the exterior derivative here has the simple action

$$\mathbf{d}\omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = \partial_{[\nu} \omega_{\mu_1 \dots \mu_p]} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \tag{A.40}$$

This can be made manifest by noting that

$$\mathbf{d} = \hat{c}^{\dagger\mu} \partial_\mu^c, \quad (\text{A.41})$$

which follows by the definition of ∂_μ^c in (A.29) and the symmetry $\Gamma_\mu^\alpha{}_\beta = \Gamma_{(\mu}^\alpha{}_{\beta)}$. (Another way to say the same is

$$\begin{aligned} \{\mathbf{d}, \hat{c}^{\dagger\mu}\} &= 0 \\ \Leftrightarrow \{\mathbf{d}, \hat{c}^{\dagger a}\} &= -\hat{c}^{\dagger\mu} \omega_\mu{}^a{}_b \hat{c}^{\dagger b}. \end{aligned} \quad (\text{A.42})$$

The second line is known as the *first structure equation* for vanishing torsion.) Therefore \mathbf{d} is *nilpotent*:

$$\begin{aligned} \mathbf{d}^2 &= \hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu} \partial_\mu^c \partial_\nu^c \\ &= 0. \end{aligned} \quad (\text{A.43})$$

(Using instead the covariant derivative shows that $\hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu} [\hat{\nabla}_\mu, \hat{\nabla}_\nu] = 0$ and hence (cf. (A.28)) $\mathbf{R}_{[\mu\nu]} = 0$.)

Furthermore it obviously satisfies the *graded Leibniz rule*:

$$\mathbf{d} \hat{c}^{\dagger\mu_1} \dots \hat{c}^{\dagger\mu_p} \omega_{\mu_1 \dots \mu_p} = \hat{c}^{\dagger\mu} \hat{c}^{\dagger\mu_1} \dots \hat{c}^{\dagger\mu_p} (\partial_\mu \omega_{\mu_1 \dots \mu_p}) + (-1)^p \hat{c}^{\dagger\mu_1} \dots \hat{c}^{\dagger\mu_p} \omega_{\mu_1 \dots \mu_p} \quad (\text{A.44})$$

This makes it easy to compute its adjoint: Let β be any p -form and α any $D-p$ -form then

$$\begin{aligned} \langle \mathbf{d}\alpha | \beta \rangle &= \int (\mathbf{d}\alpha) \wedge \star \beta \\ &\stackrel{(A.44)}{=} -(-1)^{D-p} \int \alpha \wedge \mathbf{d} \star \beta \\ &\stackrel{(A.18)}{=} -i^{-D(D-1)/2-s} (-1)^{p(p-1)/2} \int \alpha \wedge \bar{\star} \bar{\star} \mathbf{d} \bar{\star} \beta \\ &\stackrel{(A.18)}{=} - \int \alpha \wedge \star \bar{\star} \mathbf{d} \bar{\star} \beta \\ &= - \langle \alpha | \bar{\star} \mathbf{d} \bar{\star} \beta \rangle. \end{aligned} \quad (\text{A.45})$$

Hence

$$\mathbf{d}^\dagger = -\bar{\star} \mathbf{d} \bar{\star}. \quad (\text{A.46})$$

Using (A.17) this gives explicitly

$$\mathbf{d}^\dagger = -\hat{c}^\mu \hat{\nabla}_\mu. \quad (\text{A.47})$$

We will mostly refer to this “inner” derivative as the *exterior co-derivative*. It acts on $p > 0$ -forms as the covariant divergence:

$$\mathbf{d}^\dagger \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = -p \left(\nabla_\mu \omega^\mu{}_{\alpha_2 \dots \alpha_p} \right) dx^{\alpha_2} \wedge \dots \wedge dx^{\alpha_p}. \quad (\text{A.48})$$

The exterior co-derivative, being the adjoint of a nilpotent operator, is itself nilpotent:

$$\mathbf{d}^{\dagger 2} = 0. \quad (\text{A.49})$$

It is obvious, that

$$\begin{aligned} [\hat{N}, \mathbf{d}] &= \mathbf{d} \\ [\hat{N}, \mathbf{d}^{\dagger}] &= -\mathbf{d}^{\dagger}. \end{aligned} \quad (\text{A.50})$$

A.3 Dirac, Laplace-Beltrami, and spinors

The operator

$$\begin{aligned} \mathbf{D}_{\pm} &:= \mathbf{d} \pm \mathbf{d}^{\dagger} \\ &= \hat{\gamma}_{\mp}^{\mu} \hat{\nabla}_{\mu}, \end{aligned} \quad (\text{A.51})$$

is called the *Dirac operator* on $\Omega(\mathcal{M})$. Its square

$$\begin{aligned} \pm \Delta &:= \mathbf{D}_{\pm}^2 \\ &= (\mathbf{d} \pm \mathbf{d}^{\dagger})^2 \\ &= \pm \{\mathbf{d}, \mathbf{d}^{\dagger}\} \end{aligned} \quad (\text{A.52})$$

is known as the *Laplace-Beltrami operator*, which explicitly reads

$$\begin{aligned} \Delta &= \mathbf{D}_+^2 \\ &= - \left(g^{\mu\mu'} \nabla_{\mu} \nabla_{\mu'} + \Gamma_{\mu}^{\mu'\mu} \nabla_{\mu'} - R_{\mu\mu'\kappa\lambda} e^{\dagger\mu} e^{\dagger\kappa} e^{\nu} e^{\lambda} - R_{\mu\lambda} e^{\dagger\mu} e^{\lambda} \right). \end{aligned} \quad (\text{A.53})$$

This expression is known as the *Weitzenböck formula* (*cf.* for instance [30], p.130, or [3], eqs. (4.33),(4.45)). The Dirac and Laplace-Beltrami operators obviously satisfy

$$\begin{aligned} (\mathbf{D}_{\pm})^{\dagger} &= \pm \mathbf{D}_{\pm} \\ \Delta^{\dagger} &= \Delta. \end{aligned} \quad (\text{A.54})$$

Spinors. The following briefly indicates some aspects concerning spinors as viewed from the exterior geometry perspective, and how our algebraic notation relates to the more commonly used matrix representations.

The Clifford bivectors $\frac{1}{2}\hat{\gamma}_{\pm}^{ab} := \frac{1}{2}\hat{\gamma}_{\pm}^{[a}\hat{\gamma}_{\pm}^{b]}$ form a representation of the Lie algebra $\mathfrak{so}(d-s, s)$ and generate the *spin group* of the Clifford algebra, whose elements are of the form

$$R_{\pm} = \exp(\rho_{[ab]}\hat{\gamma}_{\pm}^{ab}). \quad (\text{A.55})$$

A Clifford element of the form $\psi_{\pm} = \rho R_{\pm}$, with ρ a scalar, is sometimes called a *Dirac-Hestenes state* (e.g. [31]). Applying R_{\pm} to a primitive projector P of the Clifford algebra yields the spinor representation $\psi_{\pm}P$ of the group $\text{SO}(d-s, s)$.

Now the exterior bundle can be viewed as the product of two spinor bundles. The spin groups of the two Clifford algebras $\hat{\gamma}_\pm$ act, respectively, from the left and from the right on the Clifford elements associated with an element of the exterior bundle:

This is easily seen by considering, as in (A.6), the Clifford-representation of an arbitrary (inhomogeneous) form $\boldsymbol{\omega} = \omega_{(0)} + \omega_\mu dx^\mu + \cdots = \underbrace{(\omega_{(0)} + \omega_\mu \hat{\gamma}_\pm^\mu + \cdots)}_{:=\Omega_\pm} |0\rangle$

and acting on it with the generators $\hat{\gamma}_\pm^{ab}$ of the two commuting spinor groups:

$$\begin{aligned}\hat{\gamma}_\mp^{ab} \Omega_\pm |0\rangle &= \Omega_\pm \hat{\gamma}_\mp^{ab} |0\rangle \\ &= \Omega_\pm \hat{\gamma}_\pm^{ab} |0\rangle .\end{aligned}\tag{A.56}$$

In this sense one of $\hat{\gamma}_\pm^{ab}$ acts from the left, the other from the right on the symbol map pre-images of an element of the exterior bundle.

To make this more explicit consider elements of $\Gamma(\Lambda(\mathcal{M}))$ of the form

$$\boldsymbol{\omega}_\pm := \psi_\pm \hat{O} \tilde{\psi}_\pm |0\rangle ,\tag{A.57}$$

where \hat{O}_\pm is a constant \pm -Clifford element:

$$\begin{aligned}\hat{O}_\pm &= \in \text{Cl}_\pm \\ [\partial_\mu, \hat{O}_\pm] &= 0 ,\end{aligned}\tag{A.58}$$

and where $\tilde{\cdot}$ is the linear operation of *Clifford reversion* which reverses the order of Clifford generators and takes the complex conjugate of the coefficient:

$$\tilde{(\rho \hat{\gamma}_\pm^{a_1 a_2 \cdots a_p})} := \rho^* \hat{\gamma}_\pm^{a_p \cdots a_2 a_1} .\tag{A.59}$$

Acting on such a state with a spin group element R_\pm gives

$$\begin{aligned}R_\pm \boldsymbol{\omega}_\pm &= (R_\pm \psi_\pm) \hat{O} \tilde{\psi}_\pm |0\rangle \\ \tilde{R}_\mp \boldsymbol{\omega}_\pm &= \psi_\pm \hat{O} (\widetilde{R_\mp \psi_\pm}) |0\rangle .\end{aligned}\tag{A.60}$$

To see how this goes together with the usual way of writing spinors as represented on some vector space note that

$$R_\pm \hat{\gamma}_\pm^a \tilde{R}_\pm = \Lambda_b^a \hat{\gamma}_\pm^b ,\tag{A.61}$$

as usual. By the cyclic property (A.9)

$$\begin{aligned}\Lambda^{ab} &= \langle 0 | \psi_\pm \hat{\gamma}_\pm^a \tilde{\psi}_\pm \hat{\gamma}_\pm^b | 0 \rangle_{\text{loc}} \\ &= \langle 0 | \tilde{\psi}_\pm \hat{\gamma}_\pm^b \psi_\pm \hat{\gamma}_\pm^a | 0 \rangle_{\text{loc}}\end{aligned}\tag{A.62}$$

this implies

$$\tilde{\psi}_\pm \hat{\gamma}_\pm^a \psi_\pm = \hat{\gamma}_\pm^b \Lambda_b^a .\tag{A.63}$$

Hence the construction (A.57) produces the differential forms

$$\begin{aligned}
\psi \hat{\gamma}_{\pm}^{a_1 \dots a_p} \tilde{\psi}_{\pm} |0\rangle &= \psi \hat{\gamma}_{\pm}^{a_1} \tilde{\psi}_{\pm} \dots \psi \hat{\gamma}_{\pm}^{a_p} \tilde{\psi}_{\pm} \dots |0\rangle \\
&= \Lambda^{a_1}_{[b_1} \dots \Lambda^{a_p}_{b_p]} \hat{\gamma}_{\pm}^{b_1 \dots b_p} |0\rangle \\
&= \Lambda^{a_1}_{[b_1} \dots \Lambda^{a_p}_{b_p]} dx^{b_1} \wedge \dots \wedge dx^{b_p},
\end{aligned} \tag{A.64}$$

where we set $\rho = 1$ for brevity. Now let $\phi_{\alpha} = (R_{\pm} \phi_0)_{\alpha}$ be the usual representation of the rotor R_{\pm} as a spinor on a $2^{[d/2]}$ -dimensional vector space, then the coefficients of the above differential form are obtained by means of the usual expression:⁸

$$\begin{aligned}
\bar{\phi} \gamma_{\pm b_1 \dots b_p} \phi &= \bar{\phi}_0 \tilde{\psi}_{\pm} \gamma_{\pm b_1 \dots b_p} \psi_{\pm} \phi_0 \\
&\stackrel{(A.63)}{=} \underbrace{\bar{\phi}_0 \gamma_{\pm, a_1 \dots a_p} \phi_0}_{=\text{const}} \Lambda^{a_1}_{[b_1} \dots \Lambda^{a_p}_{b_p]},
\end{aligned} \tag{A.66}$$

where now all Clifford elements refer to their matrix representation and $\bar{\phi}$ is the Dirac adjoint of ϕ .

The covariant derivative operator (A.23) $\hat{\nabla}_{\mu} = \partial_{\mu} + \omega_{\mu ab} \hat{e}^{\dagger a} \hat{e}^b = \partial_{\mu} + \frac{1}{4} (\hat{\gamma}_+^a \hat{\gamma}_+^b - \hat{\gamma}_-^a \hat{\gamma}_-^b)$ splits into a sum of covariant derivative operators

$$\hat{\nabla}_{\mu}^{S\pm} := \partial_{\mu} \pm \frac{1}{4} \omega_{\mu ab} \hat{\gamma}_{\pm}^a \hat{\gamma}_{\pm}^b \tag{A.67}$$

that act on the two spinor bundles separately:

$$\hat{\nabla}_{\mu} (\psi_+ \hat{O}_+ \tilde{\psi}_+) |0\rangle = (\hat{\nabla}_{\mu}^S \psi) \hat{O}_+ \tilde{\psi}_+ |0\rangle + \psi_+ \hat{O}_+ (\widetilde{\hat{\nabla}_{\mu}^S \psi}) |0\rangle. \tag{A.68}$$

But such a splitting does not take place for the Dirac operator (A.51) on the exterior bundle. Due to (A.35) the Dirac operators (A.51) mix the two Clifford algebra representations $\hat{\gamma}_{\pm}$.

The equation $\mathbf{D}_{\pm} \psi = (\mathbf{d} \pm \mathbf{d}^{\dagger}) \psi = 0$ is known as the (massless, free) *Kähler equation* (see [32], §8.3). Due to the above considerations it is equivalent (up to degeneracy) to the ordinary (massless, free) Dirac equation on spinors (instead of on differential forms) only when the left (+) and right (-) Clifford algebras don't mix, which occurs only for $\omega_{abc} = 0$ if no other background fields are turned on, i.e. for a flat spacetime background. But actually in string theory a generalization

⁸Hence the component analogue of (A.68) is

$$\begin{aligned}
\nabla_{\mu} (\bar{\phi}_0 \hat{\gamma}_{+a_1 \dots a_p} \phi) &= \partial_{\mu} (\bar{\phi}_0 \hat{\gamma}_{+a_1 \dots a_p} \phi) - \omega_{\mu}{}^{b_1}{}_{a_1} (\bar{\phi}_0 \hat{\gamma}_{+b_1 \dots a_p} \phi) - \dots - \omega_{\mu}{}^{b_p}{}_{a_p} (\bar{\phi}_0 \hat{\gamma}_{+a_1 \dots b_p} \phi) \\
&= \partial_{\mu} (\bar{\phi}_0 \hat{\gamma}_{+a_1 \dots a_p} \phi) - \left(\bar{\phi}_0 \left[\frac{1}{4} \omega_{\mu ab} \hat{\gamma}_+^a \hat{\gamma}_+^b, \hat{\gamma}_{+a_1 \dots a_p} \right] \phi \right) \\
&= (\overline{\nabla_{\mu}^S \phi_0}) \hat{\gamma}_{+a_1 \dots a_p} \phi + \bar{\phi}_0 \hat{\gamma}_{+a_1 \dots a_p} \nabla_{\mu}^S \phi
\end{aligned} \tag{A.65}$$

of the operators \mathbf{D}_\pm does play the role of the Dirac operator for spinors. This is possible, because the presence of further background fields will modify \mathbf{D}_\pm in a way that cancels the spurious terms and thus restores their “chirality” (in the CFT sense) (*cf.* §2.1.4 (p.17)).

A.4 Lie derivative

From \mathbf{d} one recovers a directional derivative \mathcal{L}_v along a vector field $v = v^\mu \partial_\mu$ by performing a “contraction”:

$$\mathcal{L}_v := \{\mathbf{d}, \hat{c}_\mu v^\mu\} . \quad (\text{A.69})$$

This is the *Lie derivative* on differential forms along v . More explicitly it reads

$$\begin{aligned} \{\mathbf{d}, \hat{c}_\mu v^\mu\} &= \{\hat{c}^{\dagger\mu} \partial_\mu^c, \hat{c}_\mu v^\mu\} \\ &= v^\mu \partial_\mu^c + (\partial_\mu v^\nu) \hat{c}^{\dagger\mu} \hat{c}_\nu , \end{aligned} \quad (\text{A.70})$$

or alternatively

$$\begin{aligned} \{\mathbf{d}, \hat{e}_\mu v^\mu\} &= \{\hat{c}^{\dagger\mu} \hat{\nabla}_\mu, \hat{e}_\mu v^\mu\} \\ &= v^\mu \hat{\nabla}_\mu + (\nabla_\mu v^\nu) \hat{c}^{\dagger\mu} \hat{c}_\nu . \end{aligned} \quad (\text{A.71})$$

The form (A.70) is convenient for checking that

$$[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]} \quad (\text{A.72})$$

and

$$\begin{aligned} [\mathcal{L}_v, w^\mu \hat{c}_\mu] &= [v, w]^\mu \hat{c}_\mu \\ [\mathcal{L}_v, w_\mu \hat{c}^{\dagger\mu}] &= (\mathcal{L}_v w)_\mu \hat{c}^{\dagger\mu} , \end{aligned} \quad (\text{A.73})$$

while (A.71) is convenient for computing the adjoint:

$$\begin{aligned} (\mathcal{L}_v)^\dagger &= -\frac{1}{\sqrt{g}} \hat{\nabla}_\mu \sqrt{g} v^\mu + (\nabla_\nu v^\mu) \hat{c}^\dagger_\mu \hat{c}^\nu \\ &= -\mathcal{L}_v - (\nabla_\mu v^\mu) + 2(\nabla_{(\mu} v_{\nu)}) \hat{c}^{\dagger\mu} \hat{c}^\nu . \end{aligned} \quad (\text{A.74})$$

Obviously the Lie derivative \mathcal{L}_v is skew-self-adjoint if and only if

$$\begin{aligned} \nabla_{(\mu} v_{\nu)} &= 0 \\ \Rightarrow \nabla_\mu v^\mu &= 0 , \end{aligned} \quad (\text{A.75})$$

i.e. if and only if v is a Killing vector field:

$$(\mathcal{L}_v)^\dagger = -\mathcal{L}_v \Leftrightarrow v \text{ is Killing} . \quad (\text{A.76})$$

From its definition (A.69) and the duality relations (A.15) and (A.46) it follows furthermore that the adjoint can be expressed as

$$\mathcal{L}_v^\dagger = -\bar{\star}\mathcal{L}_v\bar{\star}. \quad (\text{A.77})$$

From this we find the equivalence

$$v \text{ is Killing} \Leftrightarrow [\mathcal{L}_v, \bar{\star}] = 0. \quad (\text{A.78})$$

Hence for a Killing vector v it follows from taking the adjoint of (A.69) that

$$\{\mathbf{d}^\dagger, \hat{c}^\dagger_\mu v^\mu\} = -\mathcal{L}_v \quad (v \text{ Killing}). \quad (\text{A.79})$$

One particular consequence is, that

$$\{v_\mu \hat{\gamma}_+^\mu, \mathbf{D}_-\} - \{v_\mu \hat{\gamma}_-^\mu, \mathbf{D}_+\} = 4\mathcal{L}_v \quad (v \text{ Killing}), \quad (\text{A.80})$$

which will be rather useful later on. For the other sign one gets

$$\{v_\mu \hat{\gamma}_+^\mu, \mathbf{D}_-\} + \{v_\mu \hat{\gamma}_-^\mu, \mathbf{D}_+\} = 2(\nabla_{[\mu} v_{\nu]}) (\hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu} + \hat{c}^\mu \hat{c}^\nu) \quad (v \text{ Killing}) \quad (\text{A.81})$$

Also note that for v Killing one has

$$[\mathcal{L}_v, v \cdot \hat{\gamma}_\pm] = 0 \quad (v \text{ Killing}). \quad (\text{A.82})$$

Another useful fact is that the partial derivative operators ∂_μ^c , defined in (A.29), are obviously (using (A.41)) Lie derivatives:

$$\partial_\mu^c = \{\mathbf{d}, \hat{c}_\mu\} \quad (\text{A.83})$$

i.e.

$$\begin{aligned} \partial_\mu^c &= \mathcal{L}_{\partial_\mu} \\ &\stackrel{(A.30)}{=} \partial_\mu - (e^a{}_\alpha \partial_\mu e^\alpha{}_b) \hat{e}^{\dagger b} \hat{e}_a. \end{aligned} \quad (\text{A.84})$$

(This is to be contrasted with the Lie derivative along an ONB basis vector $v = e_a$:

$$\mathcal{L}_{e_a} = \partial_a + 2\omega_{abc} \hat{e}^{\dagger b} \hat{e}^c.) \quad (\text{A.85})$$

If ∂_μ is a Killing Lie derivative then (according to (A.84) and (A.71)) the term $(e_{a\alpha} \partial_\mu e^\alpha{}_b)$ is antisymmetric in a and b and we can write

$$\begin{aligned} \mathcal{L}_{\partial_\mu} &= \partial_\mu + (e_{a\alpha} \partial_\mu e^\alpha{}_b) \hat{e}^{\dagger a} \hat{e}^b \quad (\partial_\mu \text{ Killing}) \\ &= \partial_\mu + \frac{1}{4} (e_{a\alpha} \partial_\mu e^\alpha{}_b) (\hat{\gamma}_+^a \hat{\gamma}_+^b - \hat{\gamma}_-^a \hat{\gamma}_-^b). \end{aligned} \quad (\text{A.86})$$

Accordingly the exterior derivative (A.41) can also be written as

$$\mathbf{d} = \hat{c}^\mu \mathcal{L}_{\partial_\mu} \quad (\text{A.87})$$

and hence with (A.46) and (A.77) the exterior coderivative can also be written as

$$\mathbf{d}^\dagger = \hat{c}^\mu \mathcal{L}_\mu^\dagger. \quad (\text{A.88})$$

There is in general no Lie derivative on spinors, but along a Killing vector field there is⁹: Rewriting (A.71) in terms of Clifford generators yields

$$\begin{aligned} \mathcal{L}_v &= v^\mu \hat{\nabla}_\mu + \frac{1}{4}(\nabla_\mu v_\nu)(\hat{\gamma}_+^\mu + \hat{\gamma}_-^\mu)(\hat{\gamma}_+^\nu - \hat{\gamma}_-^\nu) \\ &= v^\mu \hat{\nabla}_\mu + \frac{1}{4}(\nabla_\mu v_\nu) \left(\hat{\gamma}_+^\mu \hat{\gamma}_+^\nu - \hat{\gamma}_-^\mu \hat{\gamma}_-^\nu \right) + \frac{1}{4}(\nabla_\mu v_\nu) \left(\hat{\gamma}_-^\mu \hat{\gamma}_+^\nu + \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu \right). \end{aligned} \quad (\text{A.89})$$

The condition that the last term vanishes is obviously $\nabla_{(\mu} v_{\nu)} = 0$, i.e. that v is Killing. Hence in this case the Lie derivative on differential forms is

$$v \text{ Killing} \Leftrightarrow \mathcal{L}_v = v^\mu \hat{\nabla}_\mu + \frac{1}{4}(\nabla_\mu v_\nu) \left(\hat{\gamma}_+^\mu \hat{\gamma}_+^\nu - \hat{\gamma}_-^\mu \hat{\gamma}_-^\nu \right) \quad (\text{A.90})$$

and splits into two Lie derivatives

$$\mathcal{L}_v^{S^\pm} = v^\mu \hat{\nabla}_\mu^{S^\pm} \pm \frac{1}{4}(\nabla_\mu v_\nu) \hat{\gamma}_\pm^\mu \hat{\gamma}_\pm^\nu \quad (v \text{ Killing}) \quad (\text{A.91})$$

on spinors (*cf.* (A.67)). If we decree that the partial derivative operator in $\hat{\nabla}^{S^\pm}$ acts only on the (respectively) left or right spinor bundle (*cf.* the factorization (A.57)), then this allows us to succinctly write

$$\mathcal{L}_v = \mathcal{L}_v^{S^+} + \mathcal{L}_v^{S^-} \quad (\text{A.92})$$

with

$$[\mathcal{L}_v^{S^+}, \mathcal{L}_w^{S^-}] = 0, \quad (\text{A.93})$$

which, together with (A.72), implies¹⁰

$$[\mathcal{L}_v^{S^\pm}, \mathcal{L}_w^{S^\pm}] = \mathcal{L}_{[v,w]}^{S^\pm}. \quad (\text{A.94})$$

An interesting special case that plays a paramount role in the context of Lie groups is that where there exists an orthonormal frame in which the Killing vector v has constant components $v^s = \delta_a^s$. In this case (A.91) gives

$$\begin{aligned} \mathcal{L}_v^S &= \hat{\nabla}_a^{S^\pm} \pm \frac{1}{4} \omega_{abc} \hat{\gamma}_\pm^b \hat{\gamma}_\pm^c \\ &= \partial_a \pm \frac{1}{2} \omega_{abc} \hat{\gamma}_\pm^b \hat{\gamma}_\pm^c \quad (\text{for } v^s = \delta_a^s). \end{aligned} \quad (\text{A.95})$$

But this is equal to the covariant derivative along v with respect to the connection with torsion $\omega'_{abc} = \omega_{abc} + T_{abc}$, where the torsion tensor $T_{abc} = \omega_{abc}$ in this frame.

⁹*cf.* p. 195 of [33]

¹⁰It should be noted, though, that (A.72) holds for arbitrary v, w , while (A.93) makes sense only for v and w both Killing, since otherwise $\hat{\nabla}_v^{S^\pm}$ isn't even defined.

A.5 Torsion.

Let

$$T_{\mu\alpha\beta} = T_{[\mu\alpha\beta]} \quad (\text{A.96})$$

be a totally antisymmetric torsion tensor, and consider the connection with torsion ω_T given by

$$\omega_{T,\mu}{}^a{}_b := \omega_\mu{}^a{}_b + T_\mu{}^a{}_b \quad (\text{A.97})$$

with the associated connection 1-form operator

$$\boldsymbol{\omega}_T{}^a{}_b := \hat{c}^{\dagger\mu} \omega_{T,\mu}{}^a{}_b, \quad (\text{A.98})$$

where ω is, as above, the (torsionless) Levi-Civita connection in the orthonormal frame. The associated covariant derivative operator is

$$\begin{aligned} \hat{\nabla}_{T,\mu} &:= \partial_\mu - (\omega_\mu{}^a{}_b + T_\mu{}^a{}_b) \hat{c}^{\dagger b} \hat{c}_a \\ &= \partial_\mu^c - (\Gamma_\mu{}^\alpha{}_\beta + T_\mu{}^\alpha{}_\beta) \hat{c}^{\dagger\beta} \hat{c}_\alpha, \end{aligned} \quad (\text{A.99})$$

whose adjoint is still of the form (A.38):

$$\hat{\nabla}_{T,\mu}^\dagger = -\frac{1}{\sqrt{|g|}} \hat{\nabla}_{T,\mu} \sqrt{|g|}. \quad (\text{A.100})$$

The operator of exterior multiplication with the torsion 2-form is

$$\mathbf{T}^\alpha := T_\mu{}^\alpha{}_\beta \hat{c}^{\dagger\mu} \hat{c}^{\dagger\beta}. \quad (\text{A.101})$$

Perturbing the exterior derivative with this operator gives

$$\begin{aligned} \mathbf{d}_T &= \mathbf{d} - \mathbf{T}^\alpha \hat{c}_\alpha \\ &= \hat{c}^{\dagger\mu} \hat{\nabla}_{T,\mu}. \end{aligned} \quad (\text{A.102})$$

Note that¹¹

$$\begin{aligned} \{\mathbf{d}, \hat{e}^{\dagger a}\} &= \mathbf{T}^a + \{\mathbf{d}_T, \hat{e}^{\dagger a}\} \\ &= \mathbf{T}^a - \boldsymbol{\omega}_T{}^a{}_b \hat{e}^{\dagger b}. \end{aligned} \quad (\text{A.103})$$

(This is the “first structure equation” in the presence of torsion, *cf.* (A.42).) Taking the adjoint gives

$$\begin{aligned} \mathbf{d}_T^\dagger &:= (\mathbf{d}_T)^\dagger = \mathbf{d}^\dagger - \mathbf{T}^{\dagger\alpha} \hat{c}^\dagger_\alpha \\ &= -\hat{c}^\mu \hat{\nabla}_{T,\mu}, \end{aligned} \quad (\text{A.104})$$

¹¹*cf.* e.g. [32], §6.4

where

$$\mathbf{T}^{\dagger\alpha} = -T_{\mu}^{\alpha} \hat{c}^{\mu} \hat{c}^{\beta}. \quad (\text{A.105})$$

The torsion-perturbed Dirac operators are

$$\begin{aligned} \mathbf{D}_{T,\pm} &:= \mathbf{d}_T \pm \mathbf{d}_T^{\dagger} = \hat{\gamma}_{\mp}^{\mu} \hat{\nabla}_{T,\mu} \\ &= \mathbf{D}_{\pm} - \hat{\gamma}_{\mp}^{\mu} T_{\mu}^{\alpha} \hat{c}^{\dagger\beta} \hat{c}_{\alpha}. \end{aligned} \quad (\text{A.106})$$

and, due to (A.104), they are still (see (A.54)) (anti-)self-adjoint:

$$(\mathbf{D}_{T,\pm})^{\dagger} = \pm \mathbf{D}_{\pm}. \quad (\text{A.107})$$

We mention some further common vocabulary associated with torsion (*cf.* §2.2 of [34]): The *Riemann curvature operator with torsion* is defined by

$$\mathbf{R}_{\mu\nu}^T = [\hat{\nabla}_{T,\mu}, \hat{\nabla}_{T,\nu}]. \quad (\text{A.108})$$

If a torsion tensor exists for which these operators and hence the Riemann curvature tensor with torsion

$$R_{\mu\nu\alpha\beta}^T = R_{\mu\nu\alpha\beta} + 2 \left(\nabla_{[\mu} T_{\nu]\alpha\beta} + T_{[\mu|\alpha\gamma|} T_{\nu]}^{\gamma}_{\beta} \right) \quad (\text{A.109})$$

vanishes, the manifold is said to be *parallelizable*. If furthermore a vielbein frame covariantly constant with respect to Γ_T exists the manifold is said to be *absolute parallelizable* (which implies ordinary parallelizability).

The associated Ricci tensor with torsion is

$$R_{\mu\nu}^T = R_{\mu\nu} - \nabla_{\alpha} T_{\mu\nu}^{\alpha} + T_{\mu\alpha\beta} T_{\nu}^{\alpha\beta}. \quad (\text{A.110})$$

The existence of a torsion making this tensor vanish is called *Ricci parallelizability*.

A.6 Conformal transformations.

Assume that the manifold \mathcal{M} is equipped with two metric tensors $g_{\mu\nu}$, $\tilde{g}_{\mu\nu}$ related by

$$\tilde{g}_{\mu\nu}(p) = e^{2\Phi(p)} g_{\mu\nu}(p) \quad (\text{A.111})$$

for some real function $\Phi : \mathcal{M} \rightarrow \mathbb{R}$. In the following all objects associated with $\tilde{g}_{\mu\nu}$ are written under a tilde, $\tilde{\cdot}$, while all other objects are associated with $g_{\mu\nu}$.

The coordinate basis forms are obviously related by

$$\begin{aligned} \tilde{\hat{c}}^{\dagger\mu} &= e^{-\Phi} \hat{c}^{\dagger\mu} \\ \tilde{\hat{c}}_{\mu} &= e^{\Phi} \hat{c}_{\mu} \end{aligned} \quad (\text{A.112})$$

and we may choose

$$\begin{aligned}\hat{e}^{\dagger a} &= \hat{e}^{\dagger a} \\ \tilde{\hat{e}}_a &= \hat{e}_a.\end{aligned}\tag{A.113}$$

Also obvious is the transformation of ∂_μ^c :

$$\tilde{\partial}_\mu^c = \partial_\mu^c + (\partial_\mu \Phi) \hat{N},\tag{A.114}$$

because this is what satisfies the definition (A.32). With (A.41) it follows that

$$\begin{aligned}\tilde{\mathbf{d}} &= \hat{c}^{\dagger\mu} \tilde{\partial}_\mu^c \\ &= e^{-\Phi} \left(\mathbf{d} + [\mathbf{d}, \Phi] \hat{N} \right).\end{aligned}\tag{A.115}$$

The conformally transformed Lie derivative operators are also readily found, for instance from (A.69):

$$\begin{aligned}\tilde{\mathcal{L}}_v &= \left\{ \tilde{\mathbf{d}}, \tilde{\hat{c}}_\mu v^\mu \right\} \\ &= \left\{ e^{-\Phi} \left(\mathbf{d} + \hat{c}^{\dagger\nu} (\partial_\nu \Phi) \hat{N} \right), e^\Phi \hat{c}_\mu v^\mu \right\} \\ &= \mathcal{L}_v + v^\mu (\partial_\mu \Phi) \hat{N}.\end{aligned}\tag{A.116}$$

The relation between the above operators and their conformal transformations is in fact a similarity transformation:

$$\begin{aligned}\hat{c}^{\dagger\mu} &= e^{-\Phi \hat{N}} \hat{c}^{\dagger\mu} e^{\Phi \hat{N}} \\ \tilde{\hat{c}}_\mu &= e^{-\Phi \hat{N}} \hat{c}_\mu e^{\Phi \hat{N}} \\ \tilde{\partial}_\mu^c &= e^{-\Phi \hat{N}} \partial_\mu^c e^{\Phi \hat{N}} \\ \tilde{\mathbf{d}} &= e^{-\Phi \hat{N}} \mathbf{d} e^{\Phi \hat{N}} \\ \tilde{\mathcal{L}}_v &= e^{-\Phi \hat{N}} \mathcal{L}_v e^{\Phi \hat{N}}.\end{aligned}\tag{A.117}$$

But this is not true for every operator:

To find $\tilde{\mathbf{d}}^\dagger$ one can for instance use (A.46) and write

$$\begin{aligned}\tilde{\mathbf{d}}^\dagger &= -\bar{*} \tilde{\mathbf{d}} \bar{*} \\ &= -\bar{*} e^{-\Phi \hat{N}} \mathbf{d} e^{\Phi \hat{N}} \bar{*} \\ &= -e^{-\Phi(D-\hat{N})} \bar{*} \mathbf{d} \bar{*} e^{\Phi(D-\hat{N})} \\ &= e^{-\Phi(D-\hat{N})} \mathbf{d}^\dagger e^{\Phi(D-\hat{N})},\end{aligned}\tag{A.118}$$

or

$$\tilde{\mathbf{d}}^\dagger = e^{-\Phi} \left(\mathbf{d}^\dagger - [\mathbf{d}^\dagger, \Phi] (D - \hat{N}) \right).\tag{A.119}$$

This is also a similarity transformation, but a different one. However, it coincides with that in (A.117) when evaluated on forms with eigenvalue n of \hat{N} equal to $n = D/2$. An immediate consequence of this result is that, for D even and when acting on forms $|\psi\rangle$ of degree $n = D/2$, the equations

$$\begin{aligned}\mathbf{d}|\psi\rangle &= 0 \\ \mathbf{d}^\dagger|\psi\rangle &= 0\end{aligned}\tag{A.120}$$

are conformally invariant in the sense that, with

$$|\tilde{\psi}\rangle := e^{-\Phi D/2} |\psi\rangle ,\tag{A.121}$$

they are equivalent to

$$\begin{aligned}\tilde{\mathbf{d}}|\tilde{\psi}\rangle &= 0 \\ \tilde{\mathbf{d}}^\dagger|\tilde{\psi}\rangle &= 0.\end{aligned}\tag{A.122}$$

A special case of this is the fact that ordinary classical source free electromagnetism in 4 dimensions is conformally invariant.

B. Proofs

Proofs of self-adjointness of the Hamiltonian In the following the proofs of the self-adjointness of various versions of the Hamiltonian generator are given.

- Ordinary case: Equation (2.85) states that

$$\mathbf{H}_{v_0}^{\dagger \hat{\eta}} = \mathbf{H}_{v_0} . \quad (\text{B.1})$$

Proof: The proof is probably easiest when using for \mathbf{H}_{v_0} the representation

$$\mathbf{H}_{v_0} = \frac{i}{2} \left(v_0 \cdot \hat{\gamma}_- \mathbf{D}_+ - v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- \right) + i\mathcal{L}_{v_0} \quad (\text{B.2})$$

(see (2.73)). Essential are furthermore the facts

$$\begin{aligned} \left\{ \mathbf{D}_+, v_0 \cdot \hat{\gamma}_+ \right\} - \left\{ \mathbf{D}_-, v_0 \cdot \hat{\gamma}_- \right\} &= \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu (\nabla_\mu v_{0\nu}) - \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu (\nabla_\mu v_{0\nu}) \\ &= 0 \end{aligned} \quad (\text{B.3})$$

(due to the antisymmetry $\nabla_\mu v_{0\nu} = \nabla_{[\mu} v_{0\nu]}$ of the covariant derivative of the Killing vector v_0) as well as

$$\begin{aligned} (i\mathcal{L}_{v_0})^\dagger &= i\mathcal{L}_{v_0} , \\ [\mathcal{L}_{v_0}, \hat{\eta}] &= 0 . \end{aligned} \quad (\text{B.4})$$

Using this, one finds

$$\begin{aligned} &\mathbf{H}_{v_0}^{\dagger \hat{\eta}} \\ &= \left(\hat{\eta} \mathbf{H}_{v_0} \hat{\eta}^{-1} \right)^\dagger \\ &= \hat{\eta} \mathbf{H}_{v_0}^\dagger \hat{\eta} \\ &\stackrel{(2.73)}{=} -\frac{i}{2} \hat{\eta} \left(v_0 \cdot \hat{\gamma}_- \mathbf{D}_+ - v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- \right)^\dagger \hat{\eta} + i\mathcal{L}_{v_0} \\ &= \frac{i}{2} \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ \left(\mathbf{D}_+ v_0 \cdot \hat{\gamma}_- - \mathbf{D}_- v_0 \cdot \hat{\gamma}_+ \right) v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ \frac{1}{v_0 \cdot v_0} + i\mathcal{L}_{v_0} \\ &= -\frac{i}{2} \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ \left(\mathbf{D}_+ v_0 \cdot \hat{\gamma}_+ - \mathbf{D}_- v_0 \cdot \hat{\gamma}_- \right) + i\mathcal{L}_{v_0} \\ &\stackrel{(B.3)}{=} \frac{i}{2} \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_- v_0 \cdot \hat{\gamma}_+ \left(v_0 \cdot \hat{\gamma}_+ \mathbf{D}_+ - v_0 \cdot \hat{\gamma}_- \mathbf{D}_- \right) + i\mathcal{L}_{v_0} \\ &= \frac{i}{2} \left(v_0 \cdot \hat{\gamma}_- \mathbf{D}_+ - v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- \right) + i\mathcal{L}_{v_0} \\ &= \mathbf{H}_{v_0} . \end{aligned} \quad (\text{B.5})$$

Stronger version:

In the special case where v_0 is covariantly constant a stronger version of this result holds.¹²

The Hamiltonian (2.73) naturally decomposes into a left and a right part

$$\mathbf{H}_{v_0}^{\text{L/R}} := \pm \frac{i}{4} [v_0 \cdot \hat{\gamma}_{\mp}, \mathbf{D}_{\pm}] . \quad (\text{B.6})$$

If v_0 is covariantly constant then these two operators are separately $\hat{\eta}$ -hermitian:

$$\left(\mathbf{H}_{v_0}^{\text{L/R}} \right)^{\dagger_{\hat{\eta}}} = \mathbf{H}_{v_0}^{\text{L/R}} \quad \text{if } \hat{\nabla}_{\mu} v_0 = 0 . \quad (\text{B.7})$$

Proof: The proof is just a special case of (B.5), making use of the fact that

$$\left\{ \mathbf{D}_{\pm}, v_0 \cdot \hat{\gamma}_{\pm} \right\} = 0 , \quad (\text{B.8})$$

which is a direct consequence of the assumption that $\hat{\nabla}_{\mu} v_0 = 0$:

$$\begin{aligned} \left(\frac{i}{4} [v_0 \cdot \hat{\gamma}_{\mp}, \mathbf{D}_{\pm}] \right)^{\dagger_{\hat{\eta}}} &= -\frac{i}{4} \hat{\eta} [v_0 \cdot \hat{\gamma}_{\mp}, \mathbf{D}_{\pm}] \hat{\eta} \\ &= \pm \frac{i}{4} \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_{\pm} [v_0 \cdot \hat{\gamma}_{\mp}, \mathbf{D}_{\pm}] v_0 \cdot \hat{\gamma}_{\pm} \\ &= \frac{i}{4} [v_0 \cdot \hat{\gamma}_{\mp}, \mathbf{D}_{\pm}] . \end{aligned} \quad (\text{B.9})$$

- k -deformed case: Equation (2.95) states that the same is true for the k -deformed Hamiltonian:

$$\mathbf{H}_{k,v_0}^{\dagger_{\hat{\eta}}} = \mathbf{H}_{k,v_0} . \quad (\text{B.10})$$

Proof: It suffices to note that the analogue of (B.3) also holds for the k -deformed case. The proof then goes through as above.

- Background b -field: In the case of a non-vanishing b -field the Hamiltonian reads

$$\mathbf{H}_{v_0} = \frac{i}{2} \left(v_0 \cdot \hat{\gamma}_{-}^{(b)} \mathbf{D}_{+}^{(b)} - v_0 \cdot \hat{\gamma}_{+}^{(b)} \mathbf{D}_{-}^{(b)} \right) + i \mathcal{L}_{v_0} . \quad (\text{B.11})$$

Equation (2.132) states that this operator is self-adjoint with respect to $\hat{\eta}^{(b)} = \frac{1}{(v_0 \hat{\gamma}_{-}^{(b)})^2} v_0 \cdot \hat{\gamma}_{-}^{(b)} v_0 \cdot \hat{\gamma}_{+}^{(b)}$.

¹²The interest in the following discussion lies in the fact that it generalizes to the case of non-vanishing b -field background.

Proof: The analogue of (B.3) is still true:

$$\begin{aligned}
& \left\{ \mathbf{D}_+^{(b)}, v_0 \cdot \hat{\gamma}_+^{(b)} \right\} - \left\{ \mathbf{D}_-^{(b)}, v_0 \cdot \hat{\gamma}_-^{(b)} \right\} \\
&= \left\{ e^{-\mathbf{W}^{(b)}} \mathbf{d} e^{\mathbf{W}^{(b)}} + e^{\mathbf{W}^{(b)\dagger}} \mathbf{d}^\dagger e^{-\mathbf{W}^{(b)\dagger}}, e^{\mathbf{W}^{(b)\dagger}} v_0 \cdot \hat{\mathbf{c}}^\dagger e^{-\mathbf{W}^{(b)\dagger}} + e^{-\mathbf{W}^{(b)}} v_0 \cdot \hat{\mathbf{c}} e^{\mathbf{W}^{(b)}} \right\} \\
&\quad - \left\{ e^{-\mathbf{W}^{(b)}} \mathbf{d} e^{\mathbf{W}^{(b)}} - e^{\mathbf{W}^{(b)\dagger}} \mathbf{d}^\dagger e^{-\mathbf{W}^{(b)\dagger}}, e^{\mathbf{W}^{(b)\dagger}} v_0 \cdot \hat{\mathbf{c}}^\dagger e^{-\mathbf{W}^{(b)\dagger}} - e^{-\mathbf{W}^{(b)}} v_0 \cdot \hat{\mathbf{c}} e^{\mathbf{W}^{(b)}} \right\} \\
&= 2 \left\{ e^{-\mathbf{W}^{(b)}} \mathbf{d} e^{\mathbf{W}^{(b)}}, e^{-\mathbf{W}^{(b)}} v_0 \cdot \hat{\mathbf{c}} e^{\mathbf{W}^{(b)}} \right\} + 2 \left\{ e^{\mathbf{W}^{(b)\dagger}} \mathbf{d}^\dagger e^{-\mathbf{W}^{(b)\dagger}}, e^{\mathbf{W}^{(b)\dagger}} v_0 \cdot \hat{\mathbf{c}}^\dagger e^{-\mathbf{W}^{(b)\dagger}} \right\} \\
&= 2\mathcal{L}_{v_0} - 2\mathcal{L}_{v_0} = 0.
\end{aligned} \tag{B.12}$$

Also, the Lie derivative along v_0 still commutes with $\hat{\eta}^{(b)}$:

$$[\mathcal{L}_{v_0}, v_0 \cdot \hat{\gamma}_\pm^{(b)}] \stackrel{[\mathcal{L}_{v_0}, \mathbf{W}^{(b)}]=0}{=} 0. \tag{B.13}$$

Therefore all ingredients are present to prove (2.132) analogously to (B.1).

Proof that \mathbf{C} weakly commutes with \mathbf{H} . The fact that the Hamiltonian generator respects the spatial constraint is proven for various cases.

- Ordinary case:

Equation (2.108) on p. 26 states that

$$[\mathbf{C}_{v_0}, \mathbf{H}_{v_0}] = \frac{1}{2iv_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ \mathbf{C}_{v_0}. \tag{B.14}$$

Proof: First we can rewrite the commutator as

$$\begin{aligned}
[\mathbf{C}_{v_0}, \mathbf{H}_{v_0}] &= [\mathbf{C}_{v_0}, -i\mathcal{L}_{v_0} + \mathbf{H}_{v_0}] \\
&= \frac{1}{4i} [v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- + v_0 \cdot \hat{\gamma}_- \mathbf{D}_+, v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- - v_0 \cdot \hat{\gamma}_- \mathbf{D}_+] \\
&= \frac{1}{2i} [v_0 \cdot \hat{\gamma}_- \mathbf{D}_+, v_0 \cdot \hat{\gamma}_+ \mathbf{D}_-] \\
&= \frac{1}{2i} (v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu (\nabla_\mu v_\nu) \mathbf{D}_- - v_0 \cdot \hat{\gamma}_+ \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu (\nabla_\mu v_\nu) \mathbf{D}_+).
\end{aligned} \tag{B.15}$$

Since \mathbf{C}_{v_0} and \mathbf{H}_{v_0} both commute with t_{v_0} this expression also commutes with t_{v_0} . This is still obvious in the third line,

$$\begin{aligned}
[[v_0 \cdot \hat{\gamma}_- \mathbf{D}_+, v_0 \cdot \hat{\gamma}_+ \mathbf{D}_-], t_{v_0}] &= \left[v_0 \cdot \hat{\gamma}_- \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_-, v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- \right] + \left[v_0 \cdot \hat{\gamma}_- \mathbf{D}_+, v_0 \cdot \hat{\gamma}_+ \frac{1}{v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_+ \right] \\
&= [-1, v_0 \cdot \hat{\gamma}_+ \mathbf{D}_-] + [v_0 \cdot \hat{\gamma}_- \mathbf{D}_+, 1] \\
&= 0.
\end{aligned} \tag{B.16}$$

but it is a nontrivial condition in the fourth line:

$$\begin{aligned}
& \left[(v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu (\nabla_\mu v_\nu) \mathbf{D}_- - v_0 \cdot \hat{\gamma}_+ \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu (\nabla_\mu v_\nu) \mathbf{D}_+), t_{v_0} \right] \\
&= \frac{1}{v_0 \cdot v_0} (v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_+ - v_0 \cdot \hat{\gamma}_+ \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_-) \\
&= -\frac{1}{2} \frac{1}{v_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\nu],
\end{aligned} \tag{B.17}$$

where the last line follows from explicitly evaluating the respective terms:

$$\begin{aligned}
& (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ \\
&= \frac{1}{4} (\nabla_{[\mu} v_{\nu]}) \left(\{v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu\} + [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] \right) \left(\{\hat{\gamma}_+^\nu, v_0 \cdot \hat{\gamma}_+\} + [\hat{\gamma}_+^\nu, v_0 \cdot \hat{\gamma}_+] \right) \\
&= \frac{1}{4} (\nabla_{[\mu} v_{\nu]}) \left(-2v_0^\mu + [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] \right) \left(2v_0^\nu + [\hat{\gamma}_+^\nu, v_0 \cdot \hat{\gamma}_+] \right) \\
&= \frac{1}{4} (\nabla_{[\mu} v_{\nu]}) \left(2v_0^\mu [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu] - 2v_0^\mu [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] - [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu] \right) \\
& \\
& (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_+ \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu v_0 \cdot \hat{\gamma}_- \\
&= \frac{1}{4} (\nabla_{[\mu} v_{\nu]}) \left(\{v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu\} + [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu] \right) \left(\{\hat{\gamma}_-^\nu, v_0 \cdot \hat{\gamma}_-\} + [\hat{\gamma}_-^\nu, v_0 \cdot \hat{\gamma}_-] \right) \\
&= \frac{1}{4} (\nabla_{[\mu} v_{\nu]}) \left(2v_0^\mu + [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu] \right) \left(-2v_0^\nu + [\hat{\gamma}_-^\nu, v_0 \cdot \hat{\gamma}_-] \right) \\
&= \frac{1}{4} (\nabla_{[\mu} v_{\nu]}) \left(2v_0^\mu [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu] - 2v_0^\mu [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] + [v_0 \cdot \hat{\gamma}_-, \hat{\gamma}_-^\mu] [v_0 \cdot \hat{\gamma}_+, \hat{\gamma}_+^\mu] \right) \tag{B.18}
\end{aligned}$$

We will now use the fact that (B.17) vanishes to prove (B.14): From the definitions

$$\begin{aligned}
\mathbf{C}_{v_0} &:= v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- + v_0 \cdot \hat{\gamma}_- \mathbf{D}_+ \\
4(\mathcal{L}_{v_0} + i\mathbf{H}_{v_0}) &:= v_0 \cdot \hat{\gamma}_+ \mathbf{D}_- - v_0 \cdot \hat{\gamma}_- \mathbf{D}_+
\end{aligned} \tag{B.19}$$

it follows that

$$\begin{aligned}
\mathbf{D}_- &= \frac{1}{2v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_+ (\mathbf{C}_{v_0} + 4(\mathcal{L}_{v_0} + i\mathbf{H}_{v_0})) \\
\mathbf{D}_+ &= -\frac{1}{2v_0 \cdot v_0} v_0 \cdot \hat{\gamma}_- (\mathbf{C}_{v_0} - 4(\mathcal{L}_{v_0} + i\mathbf{H}_{v_0})) .
\end{aligned} \tag{B.20}$$

Using this to replace \mathbf{D}_\pm in (B.15) gives the desired result:

$$\begin{aligned}
& [\mathbf{C}_{v_0}, \mathbf{H}_{v_0}] \\
&= \frac{1}{2i} \left(v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu (\nabla_{[\mu} v_{\nu]}) \mathbf{D}_- - v_0 \cdot \hat{\gamma}_+ \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu (\nabla_{[\mu} v_{\nu]}) \mathbf{D}_+ \right) \\
&\stackrel{(B.20)}{=} \frac{1}{4iv_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) \left(v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ (\mathbf{C}_{v_0} + 4(\mathcal{L}_{v_0} + i\mathbf{H}_{v_0})) + v_0 \cdot \hat{\gamma}_+ \hat{\gamma}_+^\mu \hat{\gamma}_-^\nu v_0 \cdot \hat{\gamma}_- (\mathbf{C}_{v_0} - 4(\mathcal{L}_{v_0} + i\mathbf{H}_{v_0})) \right) \\
&\stackrel{(B.17)}{=} \frac{1}{2iv_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ \mathbf{C}_{v_0}
\end{aligned}$$

- k -deformed case:

Equation (B.22) states the analogous relation for the k -deformed operators:

$$[\mathbf{C}_{k,v_0}, \mathbf{H}_{k,v_0}] = \frac{1}{2iv_0 \cdot v_0} (\nabla_{[\mu} v_{\nu]}) v_0 \cdot \hat{\gamma}_- \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu v_0 \cdot \hat{\gamma}_+ \mathbf{C}_{k,v_0} . \tag{B.22}$$

Proof: Because

$$\left\{ \mathbf{D}_{k,\pm}, v \cdot \hat{\gamma}_{\pm} \right\} = \hat{\gamma}_{\mp}^{\mu} \hat{\gamma}_{\pm}^{\nu} (\nabla_{\mu} v_{\nu} u), \quad (\text{B.23})$$

just as in the undeformed case, the proof completely parallels that given above.

C. Example: Parameter evolution in classical electromagnetism

As an example of the general constructions in §2.2.1 (p.21) we demonstrate how the Hamiltonian \mathbf{H} and the spatial constraint \mathbf{C} (2.104) look like in the special case where $\mathbf{D}_{\pm}\omega = 0$ are the Maxwell equations of sourceless classical electromagnetism.

The Faraday 2-form is

$$\begin{aligned} \mathbf{F} &= \mathbf{d}A \\ &= \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \\ &= \left(\nabla A_0 - \dot{A} \right)_i dx^i \wedge dx^0 + (\text{rot} A)_j \frac{1}{2} \epsilon^j_{kl} dx^k \wedge dx^l \\ &= \mathbf{E} \wedge dt + \mathbf{B} \\ &= E_i dx^i \wedge dx^0 + B_i \frac{1}{2} \epsilon^i_{jk} dx^j \wedge dx^k. \end{aligned} \quad (\text{C.1})$$

For Minkowski space $g = \eta$ its dual reads

$$\star \mathbf{F} = E_i \frac{1}{2} \epsilon^i_{jk} dx^j \wedge dx^k + B_i dx^i \wedge dx^0. \quad (\text{C.2})$$

The constraints $\mathbf{d}F = 0 = \mathbf{d}^{\dagger} F$ hence give

$$\begin{aligned} 0 &= \mathbf{d}\mathbf{F} \\ &= \partial_j E_i dx^j \wedge dx^i \wedge dx^0 + \partial_0 B_i \frac{1}{2} \epsilon^i_{jk} dx^0 \wedge dx^j \wedge dx^k + \partial_i B^i dx^1 \wedge dx^2 \wedge dx^3 \\ &= \left(\text{rot} E + \dot{B} \right)_j \frac{1}{2} \epsilon^j_{kl} dx^k \wedge dx^l \wedge dx^0 + (\text{div} B) dx^1 \wedge dx^2 \wedge dx^3 \\ 0 &= \mathbf{d} \star \mathbf{F} \\ &= \left(\dot{E} - \text{rot} B \right)_j \frac{1}{2} \epsilon^j_{kl} dx^k \wedge dx^l \wedge dx^0 + (\text{div} E) dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \quad (\text{C.3})$$

the components of which are the Maxwell equations.

The vector

$$v_0 = \partial_0 \quad (\text{C.4})$$

is a timelike Killing vector on Minkowski space time. The associated Clifford element is

$$v_0 \cdot \hat{\gamma}_{\pm} = -\hat{\gamma}_{\pm}^0, \quad (\text{C.5})$$

and the Hamiltonian generator (2.73) along v_0 is

$$\mathbf{H}_{v_0} = \frac{i}{2} \left((-\hat{\gamma}_-^0) \hat{\gamma}_-^i \partial_i - (-\hat{\gamma}_+^0) \hat{\gamma}_+^i \partial_i \right). \quad (\text{C.6})$$

Its action on 2-forms is given by:

$$\begin{aligned} \mathbf{H}_{v_0} \mathbf{F} &= \frac{i}{2} \left((-\hat{\gamma}_-^0) \hat{\gamma}_-^i \partial_i - (-\hat{\gamma}_+^0) \hat{\gamma}_+^i \partial_i \right) F_{\mu\nu} \hat{\gamma}_-^\mu \hat{\gamma}_-^\nu |0\rangle \\ &= -\frac{i}{2} \partial_i F_{\mu\nu} \left[\hat{\gamma}_-^0 \hat{\gamma}_-^i, \hat{\gamma}_-^\mu \hat{\gamma}_-^\nu \right] |0\rangle \\ &= -\frac{i}{2} \partial_i E_j \left[\hat{\gamma}_-^0 \hat{\gamma}_-^i, \hat{\gamma}_-^j \hat{\gamma}_-^0 \right] |0\rangle - \frac{i}{2} \partial_i B_j \frac{1}{2} \epsilon^j_{kl} \left[\hat{\gamma}_-^0 \hat{\gamma}_-^i, \hat{\gamma}_-^k \hat{\gamma}_-^l \right] |0\rangle \\ &= -i (\text{rot} E)_k \frac{1}{2} \epsilon^k_{lm} \hat{\gamma}_-^l \hat{\gamma}_-^m |0\rangle + i \partial_i B_j \epsilon^{ji}_l \hat{\gamma}_-^0 \hat{\gamma}_-^l |0\rangle \\ &= -i (\text{rot} E)_k \frac{1}{2} \epsilon^k_{lm} \hat{\gamma}_-^l \hat{\gamma}_-^m |0\rangle + i (\text{rot} B)_j \hat{\gamma}_-^j \hat{\gamma}_-^0 |0\rangle. \end{aligned} \quad (\text{C.7})$$

Therefore the evolution equation (2.72) is here equivalent to the two Maxwell equations which contain time derivatives:

$$\begin{aligned} i\mathcal{L}_{v_0} \mathbf{F} &= \mathbf{H}_{v_0} \mathbf{F} \\ \Leftrightarrow \dot{\mathbf{E}} \wedge dt + \dot{\mathbf{B}} &= (\text{rot} \mathbf{B}) \wedge dt - \text{rot} \mathbf{E}, \end{aligned} \quad (\text{C.8})$$

while the spatial constraint (2.104) is equivalent to the remaining two Maxwell equations:

$$\begin{aligned} 0 &= \mathbf{C}_{v_0} \mathbf{F} \\ &= - \left(\hat{\gamma}_-^0 \hat{\gamma}_-^i \partial_i + \hat{\gamma}_+^0 \hat{\gamma}_+^i \partial_i \right) F_{\mu\nu} \hat{\gamma}_-^\mu \hat{\gamma}_-^\nu |0\rangle \\ &= -\partial_i F_{\mu\nu} \left\{ \hat{\gamma}_-^0 \hat{\gamma}_-^i, \hat{\gamma}_-^\mu \hat{\gamma}_-^\nu \right\} |0\rangle \\ &= -\partial_i E_j \left\{ \hat{\gamma}_-^0 \hat{\gamma}_-^i, \hat{\gamma}_-^j \hat{\gamma}_-^0 \right\} |0\rangle - \partial_i B_j \frac{1}{2} \epsilon^j_{kl} \left\{ \hat{\gamma}_-^0 \hat{\gamma}_-^i, \hat{\gamma}_-^k \hat{\gamma}_-^l \right\} |0\rangle \\ &= 2\partial_i E^i |0\rangle - \partial_i B_j \epsilon^j_{kl} \hat{\gamma}_-^0 \hat{\gamma}_-^i \hat{\gamma}_-^k \hat{\gamma}_-^l |0\rangle \\ &= 2\partial_i E^i |0\rangle - \partial_i B^i \epsilon_{jkl} \hat{\gamma}_-^0 \hat{\gamma}_-^j \hat{\gamma}_-^k \hat{\gamma}_-^l |0\rangle \\ &= 2 (\text{div} E) - 6 (\text{div} B) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \end{aligned} \quad (\text{C.9})$$

When the 2-forms in 4-dimensional Minkowski space are represented as 6-dimensional column vectors, the set of equations (2.104) is therefore precisely the well known evolution equation of electrodynamics, as for instance discussed in [35].

The construction of the scalar product in §2.2.2 (p.23) also reproduces well known facts when applied to classical electromagnetism. In particular the tensor $T^{\mu\nu}$ in equation (2.97) becomes the Maxwell stress-energy tensor:

$$T^{\mu\nu} = \frac{1}{2} \langle \mathbf{F} | \hat{\gamma}_-^\nu \hat{\gamma}_+^\mu | \mathbf{F} \rangle_{\text{loc}}$$

$$\begin{aligned}
&= -\frac{1}{2} \langle 1 | \mathbf{F} \hat{\gamma}^\mu_- \mathbf{F} \hat{\gamma}^\nu_- | 1 \rangle_{\text{loc}} \\
&= F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{4} g^{\mu\nu} F_{\gamma\lambda} F^{\gamma\lambda}.
\end{aligned} \tag{C.10}$$

D. Lie groups and algebras.

Some well known relations are assembled below for references in the main text. We mostly follow the notation in §11.4 of [36].

The Lie algebra generators T_a , satisfying

$$[T_a, T_b] = f_a{}^c{}_b T_c \tag{D.1}$$

and

$$\begin{aligned}
&[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \\
&\Leftrightarrow f_{[a}{}^e{}_{|d|} f_{b}{}^d{}_{|c]} = 0
\end{aligned} \tag{D.2}$$

are represented on themselves by the adjoint action

$$\text{ad} T_a(T_b) = [T_a, T_b] = f_a{}^c{}_b T_c \tag{D.3}$$

with coefficient matrices

$$\text{ad}(T_a)^c{}_b = f_a{}^c{}_b. \tag{D.4}$$

The Killing form serves as the metric tensor

$$\begin{aligned}
\eta_{ab} &= -\frac{1}{2g^\vee} \text{tr}(\text{ad} T_a \text{ad} T_b) \\
&= -\frac{1}{2g^\vee} \text{tr}(f_a{}^c{}_s f_b{}^s{}_d) \\
&= -\frac{1}{2g^\vee} f_a{}^t{}_s f_b{}^s{}_t,
\end{aligned} \tag{D.5}$$

where g^\vee is the dual Coxeter number.

By left- and right-translation the T_a generate two commuting vielbein fields (*cf.* [12], §4.2) $e_a^\pm = e_a^{\pm\mu} \partial_\mu$:

$$\begin{aligned}
e_a^+ &= g^{-1}(\partial_a g) \\
e_a^- &= (\partial_a g) g^{-1}.
\end{aligned} \tag{D.6}$$

By default we refer to the $(-)$ vielbein when the index is omitted:

$$T_a := e_a := e_a^-. \tag{D.7}$$

The Levi-Civita connection in this basis is (*cf.* [3], (4.70))

$$\omega_{abc} = \frac{1}{2}f_{abc}. \quad (\text{D.8})$$

Therefore the covariant derivative operator (A.35) and the spinor version (A.67) read

$$\begin{aligned} \hat{\nabla}_a &= T_a + \omega_{abc}\hat{e}^{\dagger b}\hat{e}^c \\ &= T_a + \frac{1}{2}f_{abc}\hat{e}^{\dagger b}\hat{e}^c \\ \hat{\nabla}_a^{S\pm} &= T_a \pm \frac{1}{4}\omega_{abc}\hat{\gamma}_{\pm}^b\hat{\gamma}_{\pm}^c \\ &= T_a \pm \frac{1}{8}f_{abc}\hat{\gamma}_{\pm}^b\hat{\gamma}_{\pm}^c, \end{aligned} \quad (\text{D.9})$$

and the spinor Lie derivatives (A.91) along the group's Killing vectors (the vielbein components) are

$$\mathcal{L}_a^{S\pm} = \partial_a \pm \frac{1}{2}\omega_{abc}\hat{\gamma}_{\pm}^a\hat{\gamma}_{\pm}^b. \quad (\text{D.10})$$

Since the connection terms satisfy

$$\left[\frac{1}{4}f_{ast}\hat{\gamma}_{\pm}^s\hat{\gamma}_{\pm}^t, \frac{1}{4}f_{bqr}\hat{\gamma}_{\pm}^q\hat{\gamma}_{\pm}^r \right] = \pm f_a{}^c{}_b \left(\frac{1}{4}f_{csr}\hat{\gamma}_{\pm}^s\hat{\gamma}_{\pm}^r \right) \quad (\text{D.11})$$

these manifestly represent the group's Lie algebra:

$$[\mathcal{L}_a^{S\pm}, \mathcal{L}_b^{S\pm}] = f_a{}^c{}_b \mathcal{L}_c^{S\pm}. \quad (\text{D.12})$$

The spinor Lie derivatives along the invariant Killing vectors of the group manifold have the following commutators with various other objects.

$$\begin{aligned} [\mathcal{L}_a^{S+}, \hat{\gamma}_{+b}] &= f_a{}^c{}_b \hat{\gamma}_{+c} \\ \left[\mathcal{L}_a^{S+}, \frac{1}{2}\omega_{bst}\hat{\gamma}_+^s\hat{\gamma}_+^t \right] &= f_a{}^c{}_b \frac{1}{2}\omega_{cst}\hat{\gamma}_+^s\hat{\gamma}_+^t \\ [\mathcal{L}_a^{S+}, \partial_b] &= f_a{}^c{}_b \partial_c. \end{aligned} \quad (\text{D.13})$$

By the same argument familiar from the construction of the quadratic Casimir, with any two objects A_a, B_a that tranform this way an invariant under the action of \mathcal{L}_a^{S+} can be constructed:

$$\begin{aligned} [\mathcal{L}_a^{S+}, g^{bc}A_bB_c] &= g^{bc}f_a{}^d{}_b A_d B_c + g^{bc}f_a{}^d{}_c A_b B_d \\ &= 0. \end{aligned} \quad (\text{D.14})$$

The Riemann curvature operator on the group manifold is (*cf.* [22], (A.26))

$$\begin{aligned} \mathbf{R}_{ab} &= \frac{1}{4}f_{abs}f^s{}_{cd}\hat{e}^{\dagger c}\hat{e}^d \\ &= \omega_{abs}\omega^s{}_{cd}\hat{e}^{\dagger c}\hat{e}^d. \end{aligned} \quad (\text{D.15})$$

And the Ricci tensor and curvature scalar are

$$\begin{aligned}
R_{ab} &= \frac{1}{4} f_{ars} f_b{}^{rs} \\
&= \omega_{ars} \omega_b{}^{rs} \\
&\stackrel{(D.5)}{=} \frac{g^v}{2} g_{ab}
\end{aligned} \tag{D.16}$$

and

$$R = \frac{g^v d}{2} . \tag{D.17}$$

The two invariant vielbein fields e^\pm on the group manifold are characterized by the property that they are parallel with respect to the metric compatible connection with torsion $T_{\mu\nu\gamma} = \pm f_{\mu\nu\gamma}$:

$$\nabla_\mu^\pm e_a^\pm = 0 , \tag{D.18}$$

or equivalently

$$\omega[e^\pm]_{a^\pm b^\pm c^\pm} = \mp T_{a^\pm b^\pm c^\pm} . \tag{D.19}$$

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